Exercise: Find an expression for the gradient $\nabla f(x,y) = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y}$ in polar coordinates, i .e., find the functions *g* and *h* so that $\nabla f(r,\theta) = \hat{r}g(r,\theta) + \hat{\theta}h(r,\theta)$.

Solution: From the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

So we need expressions for $\partial r / \partial x,\ \partial r / \partial y,\ \ \partial \theta / \partial x,\ \ \partial \theta / \partial y$. But since $r^2 = x^2 + y^2$ we have by implicit differentiation that

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$2r\frac{\partial r}{\partial y} = 2y \quad \Rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

To get the partial derivatives of θ with respect to x and y we use $\theta = \tan^{-1}(y/x)$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{1}{x + y^2/x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Therefore

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

Thus the gradient is

$$\begin{aligned}
\nabla f(x,y) &= \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} \\
&= (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) + (\hat{r}\sin\theta + \hat{\theta}\cos\theta)\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right) \\
&= \hat{r}\cos^2\theta\frac{\partial f}{\partial r} - \hat{r}\frac{\cos\theta\sin\theta}{r}\frac{\partial f}{\partial \theta} - \hat{\theta}\sin\theta\cos\theta\frac{\partial f}{\partial r} + \hat{\theta}\frac{\sin^2\theta}{r}\frac{\partial f}{\partial \theta} \\
&\quad + \hat{r}\sin^2\theta\frac{\partial f}{\partial r} + \hat{r}\frac{\sin\theta\cos\theta}{r}\frac{\partial f}{\partial \theta} + \hat{\theta}\cos\theta\sin\theta\frac{\partial f}{\partial r} + \hat{\theta}\frac{\cos^2\theta}{r}\frac{\partial f}{\partial \theta} \\
&= \hat{r}\cos^2\theta\frac{\partial f}{\partial r} + \hat{\theta}\frac{\sin^2\theta}{r}\frac{\partial f}{\partial \theta} + \hat{r}\sin^2\theta\frac{\partial f}{\partial r} + \hat{\theta}\frac{\cos^2\theta}{r}\frac{\partial f}{\partial \theta} \\
&= \hat{r}\left(\cos^2\theta\frac{\partial f}{\partial r} + \sin^2\theta\frac{\partial f}{\partial r}\right) + \hat{\theta}\left(\frac{\sin^2\theta}{r}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial \theta}\right) \\
&= \hat{r}\frac{\partial f}{\partial r}\left(\cos^2\theta + \sin^2\theta\right) + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta}\left(\sin^2\theta + \cos^2\theta\right) \\
&= \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta}
\end{aligned}$$