Newton's Method

Newton's method is a technique that can be used to find the root of an equation, i.e., to solve the equation

$$f(x) = 0$$

for *x*. Suppose we can make some sort of educated guess, say x_0. Then consider the geometry shown in the following figure. If we draw a tangent through the point (x_0, y_0) and project it down to the *x*-axis, then the intersection of the tangent line with the *x*-axis is closer to the real root than x_0. Calling this intersection point x_1, we observe that the slope of the tangent line is

$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

Cross multiplying,

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

and therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

By the same argument, the point x_2 is an even better guess at the root, and its location is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

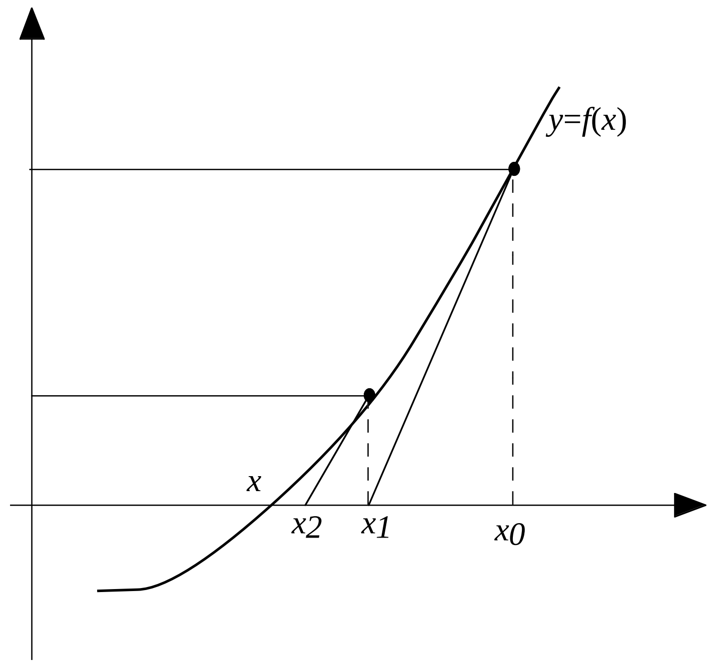
In general, we can define the sequence of guesses

$$x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots \to x \quad as \quad n \to \infty$$

where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To implement this in a computer program we could use the following algorithm:



Set $\varepsilon = 10^{-15}$ (choose any arbitrary small number)
Set $\Delta = 1$
Guess some x_0
Set $n = 0$
While $\Delta > \varepsilon$ repeat the following
  $x_{n+1} = x_n - f(x_n) / f'(x_n)$
  $n = n + 1$
  $\Delta = |x_n - x_{n-1}|$

In general this method will converge rather rapidly, usually requiring fewer than half a dozen steps to achieve an accuracy such as 10^{-15}, depending on the function.

Exercise: Write an algorithm to solve Euler's equation $M = E - e \sin E$ for *E*.

Euler’s Method for Solving Differential Equations

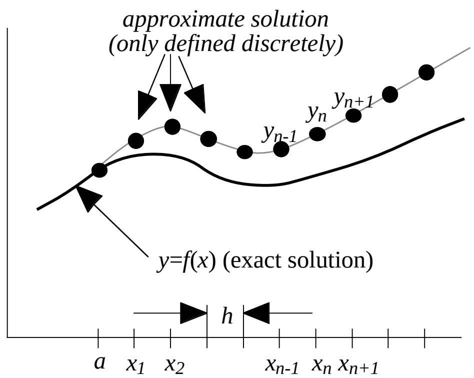
Euler’s Method addresses the problem of solving the initial value problem numerically:

$$\left.\begin{aligned} y' &= f(x,y) \\ y(a) &= y_0 \end{aligned}\right\}$$

To do this we discretize the *x*-axis into intervals of length $h > 0$,

$$x_n = a + nh, \quad n = 0,1,2,\ldots$$

as illustrated below.



While the exact solution may be defined anywhere, the approximate solution is only defined discretely at these points. We use the following notation

$y(x_n) =$ exact solution to equation (1) at $x = x_n$

$y_n =$ approximate (numerical) solution at $x = x_n$

The approximate solution y_n at $x = x_n$ is obtained in terms of difference equation relating the solutions at an adjacent set of points, e.g.,

$$\sum_{j=-k}^{m} c_j y_{n+j} = h\varphi(y_{n-k}, y_{n-k+1}, \ldots, y_{n+m}; x_n; h) \qquad \text{where}$$

φ depends on *f*. Usually these methods are linear. ***Euler’s method*** is obtained by considering the definition of a derivative,

$$\lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = y' = f(x,y)$$

For extremely small values of *h*,

$$\frac{y(x+h) - y(x)}{h} \approx f(x,y)$$

Multiplying through by *h*,

$$y(x+h) - y(x) \approx hf(x,y)$$

At $x = x_n$, since $x_{n+1} = x_n + h$, this leads to

$$y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n))$$

If at $x = x_n$ we approximate $y(x)$ by some estimate y_n, then one possible estimate y_{n+1} for $y(x_{n+1})$ is

$$\boxed{y_{n+1} \approx y_n + hf(x_n, y_n)}$$

This equation is called ***Euler's method***.

Example. Solve $y' = y$, $y(0)=1$ numerically on [0, 1] using Euler's method using $h=1$, $h = 0.5$, $h=0.2$ and $h = 0.1$, and compare it with the exact solution.

For this initial value problem, we have $f(x, y) = y$ so Euler's method is

$$y_{n+1} = y_n + hf_n = y_n + hy_n = y_n(1+h)$$

<u>With $h = 1$</u>. start with $x_0 = 1$, $y_0 = 1$ (given), and calculate successive approximations as

$$x_1=1,\ y_1 = (1+h)y_0 = 2$$

<u>With $h = 0.5$</u>, start with $x_0 = 1$, $y_0 = 1$; Then

$$x_1 = 0.5, \quad y_1 = (1+h)y_0 = (1+0.5)\cdot 1 = 1.5$$

$$x_2 = 1.0, \quad y_2 = (1+h)y_1 = (1+0.5)\cdot 1.5 = 2.25$$

<u>With $h = 0.2$</u>, start with $x_0 = 1$, $y_0 = 1$ Then.

$$x_1 = 0.2, \quad y_1 = (1+h)y_0 = (1+0.2)\cdot 1 = 1.2$$

$$x_2 = 0.4, \quad y_2 = (1+h)y_1 = (1+0.2)\cdot 1.2 = 1.44$$

$$x_3 = 0.6, \quad y_3 = (1+h)y_2 = (1+0.2)\cdot 1.44 = 1.728$$

$$x_4 = 0.8, \quad y_4 = (1+h)y_3 = (1+0.2)\cdot 1.728 = 2.0736$$

$$x_5 = 1.0, \quad y_5 = (1+h)y_4 = (1+0.2)\cdot 2.0736 = 2.48832$$

<u>With $h = 0.1$</u>. start with $x_0 = 1$, $y_0 = 1$; Then

$$x_1 = 0.1, \quad y_1 = (1+h)y_0 = (1+0.1)\cdot 1 = 1.1$$

$$x_2 = 0.2, \quad y_2 = (1+h)y_1 = (1+0.1)\cdot 1.1 = 1.21$$

$$x_3 = 0.3, \quad y_3 = (1+h)y_2 = (1+0.1)\cdot 1.21 = 1.331$$

$$x_4 = 0.4, \quad y_4 = (1+h)y_3 = (1+0.1)\cdot 1.331 = 1.4641$$

$$x_5 = 0.5, \quad y_5 = (1+h)y_4 = (1+0.1)\cdot 1.4641 = 1.61051$$

$$x_6 = 0.6, \quad y_6 = (1+h)y_5 = (1+0.1)\cdot 1.61051 = 1.77156$$

$$x_7 = 0.7, \quad y_7 = (1+h)y_6 = (1+0.1)\cdot 1.77156 = 1.94872$$

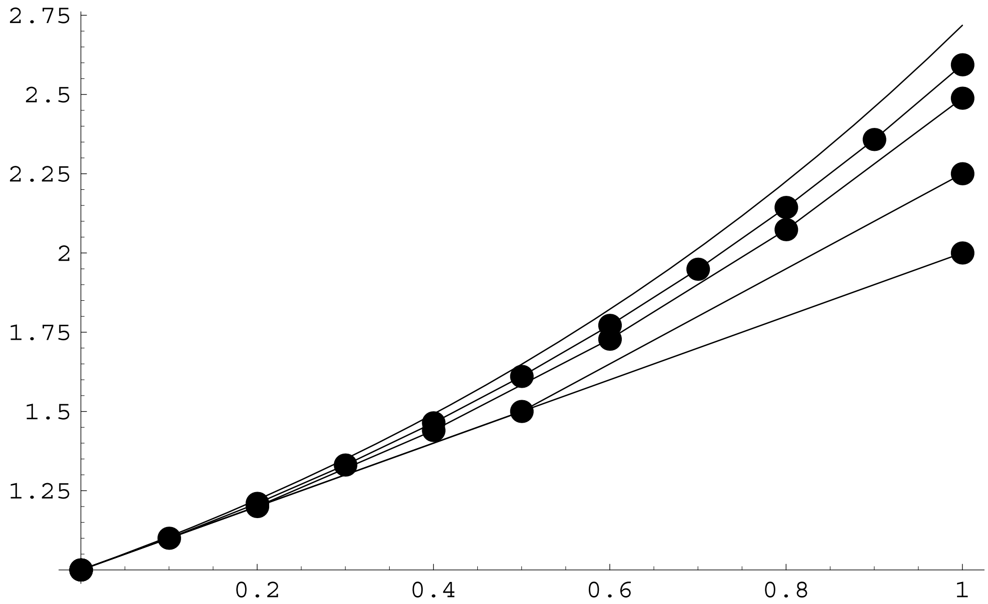
$$x_8 = 0.8, \quad y_8 = (1+h)y_7 = (1+0.1)\cdot 1.94872 = 2.14359$$

$$x_9 = 0.9, \quad y_9 = (1+h)y_8 = (1+0.1)\cdot 2.14359 = 2.35795$$

$$x_{10} = 1.0, \quad y_{10} = (1+h)y_9 = (1+0.1)\cdot 2.35795 = 2.59374$$

The exact solution to $y' = y$, $y(0)=1$ is $y = e^x$

..



The above figure illustrates the four numerical solutions.

Runge-Kutta Methods

An ***s-stage Runge-Kutta method*** is any method to solve the initial value problem $y' = f(x,y),\ y(a) = y_0$ that has the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$$

where

$$k_i = f\left(x_n + h\sum_{j=1}^{i-1} a_{ij},\quad y_n + h\sum_{j=1}^{i-1} a_{ij}k_j \right) \quad \text{for} \quad i = 1,2,\ldots,s$$

The most common Runge-Kutta method is the following 4-stage method:

$$y_{n+1} = y_n + h\left(\tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4\right)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

Consider the previous example, $y' = y$, $y(0)=1$. The general form of the Runge-Kutta solution for this differential equation is

$$k_1 = f(x_n, y_n) = y_n$$

$$k_2 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) = y_n + \tfrac{1}{2}hk_1 = y_n(1 + h/2)$$

$$k_3 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) = y_n + \tfrac{1}{2}hk_2 = y_n + \tfrac{1}{2}hy_n(1 + h/2)$$

$$= y_n(1 + h/2 + h^2/4)$$

$$k_4 = f(x_n + h, y_n + hk_3) = y_n + hk_3 = y_n + hy_n(1 + h/2 + h^2/4)$$

$$= y_n(1 + h + h^2/2 + h^3/4)$$

$$y_{n+1} = y_n + h[y_n/6 + (1/3)y_n(1 + h/2) + (1/3)y_n(1 + h/2 + h^2/4)$$

$$+(1/6)y_n(1 + h + h^2/2 + h^3/4)]$$

$$= y_n[1 + h(1/6 + 1/3 + h/6 +$$

$$1/3 + \quad + h/6 + h^2/12 +$$

$$1/6 + \quad + h/6 + h^2/12 + h^3/24)]$$

$$= y_n[1 + h(1 + h/2 + h^2/6 + h^3/24)]$$

$$= y_n[1 + h + h^2/2 + h^3/6 + h^4/24]$$

Using h = 1 and starting with $x_0 = 1$, $y_0 = 1$, we have

$$y_1 = 1(1 + 1 + 1/2 + 1/6 + 1/24) = 2.70833$$

which is closer to the exact solution at $y = 1$ ($y = e^x \Rightarrow y(1) = e \approx 2.71828$) then any of the Euler methods, with just a single step!

Numerically, it is generally more practical to calculate each of the k_i and then plug them into the formula for y_{n+1} rather than just figure out a general formula first, as we did in this example. Of course you should get the same result if you do this.