Math 481A
Due dates for specific problems will be announced in class
Last corrected: June 24, 2008

1. Show that $f'(x)$ is 0 at least once in the given intervals:
   (a) $f(x) = 1 - e^x + (e-1)\sin((\pi/2)x)$ on $[0,1]$
   (b) $f(x) = (x-1)\sin x \ln(x+2)$ on $[-1,3]$
2. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.
3. For the Taylor polynomial in problem 2, use $P_2(0.5)$ to approximate $f(0.5)$ and then use the remainder theorem to find a bound for the error $|f(0.5) - P_2(0.5)|$. Compare the predicted and actual error.
4. For the Taylor polynomial in problem 2, use the remainder theorem to find a bound for the error $|f(x) - P_2(x)|$ on the entire interval $[0,1]$.
5. Let $f(x)$ be as defined in problem 2. Approximate $\int_0^1 f(x)dx$ using $P_2(x)$ and compare it to the actual answer. Note: $\int e^x \cos x = \frac{1}{2}e^x(\cos x + \sin x)$. You do not have to derive this formula.
6. Use three-digit chopping to find the sum $\sum_{i=1}^{10} \frac{1}{i^2}$ in two ways: as the sum $\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{81} + \frac{1}{100}$ and then as the sum $\frac{1}{100} + \frac{1}{81} + \cdots + \frac{1}{4} + \frac{1}{1}$. Which is more accurate, and why?
7. The Taylor Polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n}(x^i/i!)$. Use the Taylor polynomial of degree 9 and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods:
   (a) $e^{-5} \approx \sum_{i=0}^{9} \frac{(-1)^i(5^i)}{i!}$
   (b) $e^{-5} \approx 1/\sum_{i=0}^{9} \frac{5^i}{i!}$
8. Which of the values $0.01, 0.02, 0.03, \ldots, 0.99, 1.00$ are machine numbers assuming base 2 floating point arithmetic?
9. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a *Lipshitz Condition* with Lipshitz constant l on an interval $[a,b]$ if, for every $x, y \in [a,b]$ we have

$$|f(x) - f(y)| \leq L|x-y|$$

   (a) Show that if f satisfies a Lipschitz condition with Lipschitz constant L on an interval $[a,b]$ then f is continuous on $[a,b]$.
   (b) Show that if f has a bounded derivative on $[a,b]$, where the derivative is bounded by L, then f satisfies a Lipschitz condition with Lipschitz constant L on $[a,b]$.
   (c) Give an example of a function that is continuous on a closed interval but does not satisfy a Lipshitz condition on the same interval.

10. Use the bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

11. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the bisection algorithm. (Hint: Use bisection on a function that has a root at $\sqrt{3}$).

12. Find a bound on the number of iterations required to find the solution of $x^3 - x - 1 = 0$ on $[1, 2]$ to an accuracy of 10^{-5} using the bisection algorithm

13. For the equation
$$x - \cos x = 0$$
determine a function $g(x)$ and an interval $[a, b]$ on which fixed-point iteration will converge to a positive solution of the equation, and then find the solution with an absolute error of at most 10^{-5} using fixed-point iteration.

14. Let $g(x)$ be a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that $g'(x)$ exists on $[a, b]$ and that $g(x)$ satisfies a Lipshitz condition on the interval $[a, b]$ with Lipshitz constant $L < 1$. Show that for any number $p_0 \in [a, b]$, the sequence defined by $p_n = g(p_{n-1})$ converges to the unique fixed point $p \in [a, b]$.

15. Show that if A is any positive number then the sequence defined by
$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}$$
converges to $\sqrt{A}$ for any $x_0 \in (a, \infty)$ for some value of $a > 0$.

16. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3 with
    (a) Newton's method
    (b) The secant method

17. If you invest P dollars at an interest rate of i percent (e.g., $i = .03$ means 3%) compounded monthly then after t years the amount A in your account will be
$$A = P\left(1 + \frac{i}{12}\right)^{12t}$$
    (a) Prove that if you add the same amount P dollars every month to this account then after t years you will actually have
$$A = \frac{12P}{i}\left[\left(1 + \frac{i}{12}\right)^{12t} - 1\right]$$
    Hint: this is a finite geometric sum.
    (b) Suppose that you want to save \$10,000,000 in this investment to assure yourself of a comfortable retirement 50 years from now. If you can afford to save \$100 per month, use Newton's method to determine the minimum interest rate you need for your investment to an accuracy of 3 significant figures.

18. If you invest P dollars at an interest rate i percent compounded m times per year then after t years you will have

$$A = P\left(1 + \frac{i}{m}\right)^{mt}$$

   (a) The limit as $m \to \infty$ is called the Continuous compounding of interest. Use the fact that

$$\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^{x} = e$$

   to prove that

$$A = Pe^{rt}$$

   for continuous compounding.

   (b) Suppose you invest P dollars every month for t years in an account that compounds interest continuously. Show that after t years you will have

$$A = \sum_{k=0}^{12t} Pe^{ik/12} = \frac{P(e^{rt+r/12} - 1)}{e^{r/12} - 1}$$

   (c) What interest rate do you need to have \$10,000,000 after 50 years?

19. (a) Show that for any positive integer k the sequence $p_n = 1/n^k$ converges linearly to $p = 0$.

   (b) Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to zero.

20. Suppose that $\lambda = 0.5$ and $p_n \to 0$. We found in class that if p_n converges linearly then

$$|p_n| \approx (0.5)^n |p_0|$$

and that for quadratic convergence

$$|p_n| \approx (0.5)^{2^n - 1} |p_0|^{2^n}$$

Derive a formula for cubic convergence and compare with linear and quadratic convergence by calculating $p_1, p_2, .., p_7$ (i.e., extend table 2.7 of the text, 7th edition, page 79 to include a column for cubic convergence as well as linear and quadratic).

21. A sequence p_n is said to be **superlinearly convergent** to p if

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0$$

   (a) Show that if $p_n \to p$ of order $\alpha > 1$ then p_n is superlinarly convergent to p.

   (b) Show that $p_n = 1/n^n$ is superlinearly convergent to 0.

   (c) Show that $p_n = 1/n^n$ does not converge to 0 of order α for any $\alpha > 1$.

22. Let $f(x) = x^2 - 2$.

   (a) Use the fixed point iteration theorem to show that Newton's method converges to a root of $f(x)$ on the interval $[1, 2]$.

(b) Let $p_0 = 2.0$. Use the fixed point iteration error formula to estimate the number of iterations it will take to converge to 100,000 digits of accuracy.

23. Show that Newton's Method formula for finding $\sqrt{2}$ converges quadratically by finding the limit

$$\lim_{n\to\infty} \frac{|p_{n+1} - \sqrt{2}|}{|p_n - \sqrt{2}|^2}$$

Hint: Let $f(x) = x^2 - 2$; find an expression for $g(x)$ where $x_{n+1} = g(x_n)$ according to Newton's Method. Define a variable $x = p_n$ and then find the limit

$$\lim_{x\to\sqrt{2}} \frac{|g(x) - \sqrt{2}|}{|x - \sqrt{2}|^2}$$

You need to explain why using this hint works as well as doing the calculations.

24. Let

$$g(x) = -\frac{1}{2x^3} + \frac{3}{2x} + \frac{3x}{8}$$

be an iteration formula under fixed point iteration. Show that $p_n \to \sqrt{2}$ with order 3 and find the asymptotic error constant.

25. Use Aitken's Δ^2 method to generate $\hat{p_n}$ for three iterations of $p_n = 1/n$.

26. Suppose that $f(x) = \sqrt{1+x}$. Let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct the Lagrange interpolation polynomials of degree at most 1 and 2 to approximate $f(0.45)$.

27. Use the error theorem for Lagrange polyonomials to estimate a bounds on the error in the previous problem and compare with the actual error.

28. Use the same method we applied in the proof of the error theorem for Lagrange polynomials to prove Taylor's Theorem:

Let f be $n-$times differentiable on $[a, b]$, suppose that $f^{(n+1)}$ exists on $[a, b]$, and suppose that $x_0 \in [a, b]$. Then for every $x \in [a, b]$ there exists a number ξ between x_0 and x with

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Hint: Let

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \cdot \frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}}$$

where P is the n^{th} Taylor Polynomial, and use Rolle's theorem.

29. Compute a Newton-interpolation difference table for the following data:

$$((0, 0), (1, 1/2), (2, 2/5), (3, 3/10), (4, 4/17), (5, 5/26))$$

and use it to estimate $f(1.7)$.

30. Derive a method for approximating $f'''(x_0)$ whose error term is of order h^2 by expanding the function f in a fourth-order Taylor polynomial about x_0 and evaluating it at $x_0 \pm h$ and $x_0 \pm 2h$.

31. The following data give approximations to the $I = \int_0^{3\pi/4} \cos x \, dx$

$$\begin{aligned} N_1(h) &= 2.356194 \\ N_1(h/2) &= -0.47879837 \\ N_1(h/4) &= -0.8815732 \\ N_1(h/8) &= -0.9709157 \end{aligned}$$

Use Richardson's extrapolation to construct an extrapolation table and determine $N_4(h)$, assuming that

$$I = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + K_4 h^8 + O(h^{10})$$

32. Approximate

$$\int_0^2 x^2 e^{-x} dx$$

using each of the following methods. If you use a computer, user $h = 0.1$. If you do it by hand, use $h = 0.5$.

(a) Trapezoidal Rule
(b) Simpson's Rule
(c) Midpoint Rule

33. Prove that the initial value problem

$$\frac{dy}{dt} = 1 + (t - y)^2, \ y(2) = 1$$

has a unique solution on $[2, 3]$.

34. Use Picard iteration to calculate the first three iterations ϕ_0, ϕ_1, ϕ_2 for

$$\frac{dy}{dt} = 1 + (t - y)^2, \ y(2) = 1$$

35. Use Euler's method to approximate the solution to

$$\frac{dy}{dt} = 1 + (t - y)^2$$

on $[2, 3]$ with $y(2) = 1$ and $h = 0.5$

36. Use Euler's method to solve the following system of equations:

$$\begin{aligned} x' &= y - 2z \\ y' &= z - 2x \\ z' &= x - 2y - z \end{aligned}$$

on $t \in [0, 10]$ (if you use a computer, or on $[0, 0.03]$ by hand), using $h = 0.01$ and $(x_0, y_0, z_0) = (1.0, 2.0, 1.0)$.