Exam 3 Solutions

1. Find the linearization of $f(x) = 8\sqrt{5x + \frac{20}{x}}$ about $x = 4$ in the form $y = mx + b$ and simplify.

$$f(4) = 8\sqrt{5(4) + \frac{20}{4}} = 8\sqrt{25} = (8)(5) = 40$$

$$f'(x) = (8)\left(\frac{1}{2}\right)\left(5x + \frac{20}{x}\right)^{-1/2}\left(5 - \frac{20}{x^2}\right) = \frac{4\left(5 - \frac{20}{x^2}\right)}{\sqrt{5x + \frac{20}{x}}}$$

$$f'(4) = \frac{4\left(5 - \frac{20}{4^2}\right)}{\sqrt{5(4) + \frac{20}{4}}} = \frac{4\left(5 - \frac{20}{16}\right)}{5} = \frac{4(5(16) - 20)}{5(16)} = \frac{4(80 - 20)}{80} = \frac{240}{80} = 3$$

$y = f(4) + f'(4)(x - 4) = 40 + 3(x - 4) = 40 + 12x - 12 = 12x - 28$

2. A balloon is rising at a constant speed of 5 feet/second. A boy is cycling along a straight road at a speed of 15 feet/second. When he passes under the balloon it is 45 feet above him. How fast is the distance between the boy and the baloon increasing 3 seconds later (simplify)?

Given: $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = 15$, when $x = 0$, $y = 45$

Then: at $t = 3$, $y(3) = 45 + (3)(5) = 45 + 15 = 60$

Also: at $t = 3$, $x(3) = 0 + (3)(15) = 45$

Let D be the distance from the boy to the baloon. Then $D^2 = x^2 + y^2$.

Hence at $t = 3$, $D(3)^2 = x(3)^2 + y(3)^3 = 60^2 + 45^2 = 3600 + 2025 = 5625 = (9)(625) = (3)(3)(25)(25) \implies D(3) = (3)(25) = 75$

Differentiating, $2DD' = 2xx' + 2yy' \implies \frac{dD}{dt} = \frac{xx' + yy'}{D}$

At $t = 3$, $\frac{dD}{dt} = \frac{xx' + yy'}{D} = \frac{(45)(15) + (60)(5)}{75} = \frac{675 + 300}{75} = \frac{975}{75} = 13$ feet/second.

3. The circumference of a sphere is measured to be 84 centimeters with a possible error of 0.5 centimeters. Estimate the maximum error and percentage error in the calculated surface area.

The circumference gives the radius of the sphere: $2\pi r = 84 \implies r = \frac{84}{2\pi} = \frac{42}{\pi}$

Since $C = 2\pi r \implies dC = 2\pi dr$ then the error in the radius is $dr = \frac{dC}{2\pi} = \frac{.5}{2\pi} = \frac{1}{4\pi}$

The surface area is $A = 4\pi r^2$.

Hence the error is $dA = 8\pi r dr = (8\pi)\left(\frac{42}{\pi}\right)\left(\frac{1}{4\pi}\right) = \frac{84}{\pi}$ cm^2

The percentage error in the area is

$$\frac{\text{error}}{\text{total area}} \times 100 = \frac{100dA}{A} = \frac{100 \times 8\pi r dr}{4\pi r^2} = \frac{200dr}{r} = \frac{(200)(1/4\pi)}{42/\pi} = \frac{50}{42} = \frac{25}{21}\%$$

4. Find all numbers that satisfy the mean value theorem for $f(x) = x^3 + x - 1$ on $[0, 2]$.

   Since $f(0) = -1$ and $f(2) = 2^3 + 2 - 1 = 8 + 2 - 1 = 9$ then the mean value theorem says that there is some number c between 0 and 2 such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{9 - (-1)}{2} = 5$$

   Since $f'(x) = 3x^2 + 1$ this means we must solve $3x^2 + 1 = 5$ for x.

$$3x^2 + 1 = 5 \implies 3x^2 = 4 \implies x^2 = \frac{4}{3} \implies x = \pm\frac{2}{\sqrt{3}}$$

   Since only the positive solution lies in $[0, 2]$ the only point that satisfies the mean value theorem is $x = 2/\sqrt{3}$.

5. Identify (a) the critical points and find the (b) absolute maximum and (c) absolulute minimum of $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$ on $[-2, 4]$.

   (a) The critical points are the endpoints: $x = -2$ and $x = 4$ and the points where $f' = 0$. Differentiating,

$$0 = f'(x) = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}$$

   Since the denominator is always greater than or equal to 4, it can never be zero. This means $16x = 0$ or $x = 0$ is the only critical point for $f'(x) = 0$.

   Hence the critical points are at $x = -2, x = 0, x = 4$.

   (b) and (c) To get the maximum and minimum we calculate y at each point.

   $y(-2) = \dfrac{4 - 4}{4 + 4} = 0$

   $y(0) = \dfrac{0 - 4}{0 + 4} = -1$ (Absolute Minimum at $x = 0$)

   $y(4) = \dfrac{16 - 4}{16 + 4} = \dfrac{12}{20} = 0.6$ (Absolute Maximum at $x = 4$)

6. Suppose that a ball is given a push so that the distance rolled after t seconds is $s(t) = 5t + 3t^2$. Find (a) the velocity after 2 seconds; and (b) how long does it take for the velocity to reach 35 meters/second?

   (a) Differentiating, $s'(t) = 5 + 6t$. Hence $s'(2) = 5 + (6)(2) = 17$ meters/second

   (b) Solve $s'(t) = 35$ for t.

$$35 = 5 + 6t \implies 30 = 6t \implies t = \frac{30}{6} = 5 \text{ seconds}$$

7. The volume of a cube is increasing at a rate of 10 cm^3/minute. How fast is the surface area increasing when the length of an edge is 30 cm.?

   Let the length of each side be x. Then the volume is $V = x^3$ and the area is $A = 6x^2$.

   We are given $\dfrac{dV}{dt} = 10$. Since

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \implies \frac{dx}{dt} = \frac{10}{3x^2}$$

When $x = 30$ cm., this means

$$\frac{dx}{dt} = \frac{10}{3(30)^2} = \frac{1}{270} \text{ cm/minute}$$

Differentiating $A = 6x^2$ gives

$$\frac{dA}{dt} = 12x\frac{dx}{dt} \implies \left.\frac{dA}{dt}\right|_{\text{when } x=30} = (12)(30)\left(\frac{1}{270}\right) = \frac{4}{3} \text{ cm/minute}$$

8. Estimate $\sqrt{170}$ using differentials.

   Using differentials is equivalent to using the linearization.

   Let $f(x) = \sqrt{x}$ near $x = 169$ because $f(169) = 13$.

   Then $f'(x) = \dfrac{1}{2\sqrt{x}}$.

   Then

   $$f(170) \approx f(169) + dy = 13 + dy$$

   where

   $$dy = f'(x)dx = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{169}}(170 - 169) = \frac{1}{26}$$

   Hence

   $$\sqrt{170} \approx 13 + \frac{1}{26} = \frac{339}{26} = 13\frac{1}{26}$$

9. Find the intervals where $f(x) = x^4 - 2x^3 + 3$ is increasing and decreasing.

   $y' = 4x^3 - 6x^2$

   It can change from increasing to decreasing or vice-versa when $y' = 0$, i.e., when

   $$0 = 4x^3 - 6x^2 = x^2(4x - 6) \implies x = 0, 3/2$$

   These points divide the real axis into three intervals: $(-\infty, 0)$, $(0, 3/2)$, and $(3/2, \infty)$. We pick a point in each interval to see if the function is either increasing or decreasing.

   In $(-\infty, 0)$ pick $x = -1$. $f'(-1) = (-1)^2(4(-1) - 6) < 0 \implies$ decreasing.

   In $(0, 3/2)$ pick $x = 1$. $f'(1) = 4 - 6 = -2 < 0 \implies$ decreasing.

   In $(3/2, \infty)$ pick $x = 2$. $f'(2) = 2^(4(2) - 6) > 0 \implies$ increasing.

10. Find the inflection points and identify the intervals of concavity for $f(x) = x^4 - 2x^3 + 3$.

    We have $y' = 4x^3 - 6x^2$ hence $y'' = 12x^2 - 12x$.

    Concavity can only change when $y'' = 0$ which means $0 = 12x^2 - 12x = 12x(x - 1)$. The possible inflection points are $x = 0$ and $x = 1$.

    These divide the real line into three intervals. To determine the concavity in each interval we pick a test point in each interval.

    In $(-\infty, 0)$ pick $x = -1$.

    $$f'(-1) = 12(-1)^2 - 12(-1) = 12 + 12 = 24 > 0 \implies \text{Concave Up in } (-\infty, 0)$$

In $(0,1)$ pick $x = 1/2$.

$$f'(1/2) = 12(1/2)^2 - 12(1/2) = 12/4 - 6 = 3 - 6 = -3 < 0 \implies \text{ Concave Down in } (0,1)$$

In $(1,\infty)$ pick $x = 2$.

$$f'(2) = 12(2^2) - 12(2) = 48 - 24 = 24 > 0 \implies \text{ Concave up in } (1,\infty)$$

Hence there are inflection points at $x = 0$ and $x = 1$.

For problems 9 and 10, the function looks like this. The dots are critical points and the ovals surround the inflection points.