

Math 150A – Exam 1 Answers  
Section 15608

11 Sept. 2009

1. (a)  $\lim_{x \rightarrow 1} \frac{5x^3 - 4}{3x^2 + 97} = \frac{5 - 4}{3 + 97} = \frac{1}{100}$

(b)  $\lim_{x \rightarrow 4} \frac{\frac{1}{4} - \frac{1}{x}}{x - 4} = \lim_{x \rightarrow 4} \frac{\frac{x - 4}{4x}}{x - 4} = \lim_{x \rightarrow 4} \frac{1}{4x} \rightarrow \frac{1}{16}$

(c)  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - x - 6} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{(x - 3)(x + 2)} = \lim_{x \rightarrow 3} \frac{x + 3}{x + 2} \rightarrow \frac{3 + 3}{3 + 2} = \frac{6}{5}$

(d)  $\lim_{x \rightarrow 0} \frac{7 - \sqrt{49 - x^2}}{x^2}$  - Rationalize the numerator to get:

$$\frac{7 - \sqrt{49 - x^2}}{x^2} \times \frac{7 + \sqrt{49 - x^2}}{7 + \sqrt{49 - x^2}} = \frac{x^2}{x^2(7 + \sqrt{49 - x^2})} = \frac{1}{7 + \sqrt{49 - x^2}} \rightarrow \frac{1}{14}$$

2. The precise definition of  $\lim_{x \rightarrow a} f(x) = L$  says that for any  $\epsilon > 0$ , there exists a  $\delta > 0$ , such that  $|x - a| < \delta \implies |f(x) - L| < \epsilon$ . In the remainder of this problem consider the limit

$$\lim_{x \rightarrow 4} (x^2 - 8x - 1) = -17$$

(a) (5 pts) Write the expression  $|x - a| < \delta \implies |f(x) - L| < \epsilon$  for  $\lim_{x \rightarrow 4} (x^2 - 8x - 1) = -17$

$$|x - 4| < \delta \implies |(x^2 - 8x - 1) - (-17)| < \epsilon$$

(b) (5 pts) Suppose I tell you what value of  $\epsilon$  to use. Find a value of  $\delta$  that will work.

$$|x^2 - 8x + 1 + 17| < \epsilon \implies |x^2 - 8x + 16| < \epsilon \implies |x - 4|^2 < \epsilon \implies |x - 4| < \sqrt{\epsilon}$$

This means any  $\delta \leq \sqrt{\epsilon}$  will work.

(c) (5 pts) Prove that  $\lim_{x \rightarrow 4} (x^2 - 8x - 1) = -17$

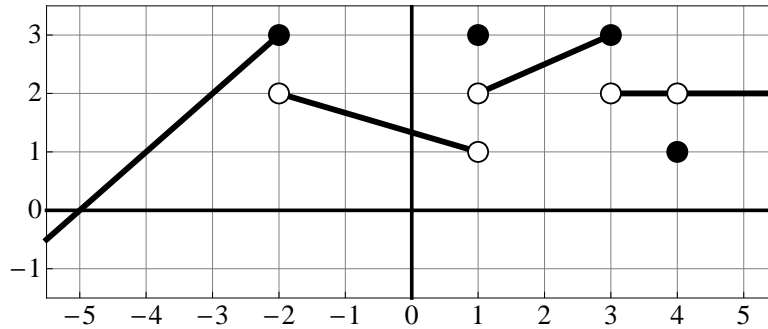
Let  $\epsilon > 0$  be given. Then define  $\delta = \sqrt{\epsilon}$ . This means that

$$\begin{aligned} |x - 4| < \delta &\implies |x - 4| < \sqrt{\epsilon} \\ &\implies |x - 4|^2 < \epsilon \\ &\implies |x^2 - 8x + 16| < \epsilon \\ &\implies |(x^2 - 8x - 1) - (-17)| < \epsilon \end{aligned}$$

(d) (5 pts) Using the result of part (b), suppose that  $\epsilon = 0.01$ , and find a value of  $\delta$ .

$$\delta = \sqrt{0.01} = 0.1$$

3. Determine each of the following values from the graph. Write DNE if a value does not exist.



(a) (1 pt)  $\lim_{x \rightarrow -3} f(x) = 2$

(e) (1 pt)  $f(-2) = 3$

(b) (1 pt)  $\lim_{x \rightarrow 3^-} f(x) = 3$

(f) (1 pt)  $f(1) = 3$

(c) (1 pt)  $\lim_{x \rightarrow 3^+} f(x) = 2$

(g) (1 pt)  $\lim_{x \rightarrow 4} f(x) = 2$

(d) (1 pt)  $\lim_{x \rightarrow 3} f(x) = \text{DNE}$

(h) (1 pt)  $\lim_{x \rightarrow 1^+} f(x) = 2$

(i) (3 pts) Write the interval where  $f(x)$  is continuous:

$$(\infty, -2) \cup (-2, 1) \cup (1, 3) \cup (3, 4) \cup (4, \infty)$$

(j) (1 pts) Write the definition of the statement “ $f(x)$  is continuous at  $x = a$ ”.

$$\lim_{x \rightarrow a} f(x) = f(a)$$

(k) (4 pts) Explain why  $f(x) = x^3$  is continuous at  $x = 2$ .

$$f(2) = 2^3 = 8, \text{ and also } \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} x^3 = 8, \text{ so } \lim_{x \rightarrow 2} f(x) = f(2).$$

(l) (4 pts) Find a value of  $c$  that will make the function  $f(x) = \begin{cases} x^3 - 3cx, & x < 3 \\ cx, & x \geq 3 \end{cases}$  continuous at  $x = 3$ .

$$(3^3) - 3c(3) = 3c$$

$$27 - 9c = 3c$$

$$27 = 12c$$

$$c = \frac{27}{12} = \frac{9}{4} = 2.25$$

4. Find an equation of the tangent line to  $y = 3x^2 - 4x$  at  $x = 3$ .

To get the line we need a point and a slope. To find a point, substitute  $x = 3$  to get:

$$y(3) = 3(3)^2 - 4(3) = 27 - 12 = 15$$

Hence  $(3, 15)$  is a point on the tangent line.

To get the slope, we need the derivative at  $x = 3$ . We already know  $f(3) = 15$ ;

$$\begin{aligned} f(3+h) &= 3(3+h)^2 - 4(3+h) = 3(9+6h+h^2) - 12 - 4h \\ &= 27 + 18h + 3h^2 - 12 - 4h = 15 + 14h + 3h^2 \\ f(3+h) - f(3) &= 15 + 14h + 3h^2 - 15 = 14h + 3h^2 = h(14+3h) \\ \frac{f(3+h) - f(3)}{h} &= \frac{h(14+3h)}{h} = 14 + 3h \\ f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} (14 + 3h) = 14 \end{aligned}$$

Alternatively,

$$\frac{f(x) - f(3)}{x - 3} = \frac{3x^2 - 4x - 15}{x - 3} = \frac{(x-3)(3x+5)}{x-3} = 3x + 5 \rightarrow 3(3) + 5 = 14 \text{ as } x \rightarrow 3$$

Therefore the slope is 14. So the line is  $y = 14x + b$

To get  $b$ , substitute the point  $(3, 15)$ :  $15 = 14(3) + b = 42 + b \implies 15 - 42 = b \implies -27 = b$

hence the equation of the line is  $y = 14x - 27$

5. Let  $f(x) = 1/x$ .

(a)(3 pts) Find the average rate of change of  $f(x)$  on the interval  $[1, 2]$ .

$$\frac{f(2) - f(1)}{2 - 1} = \frac{1/2 - 1}{1} = -1/2$$

(b)(3 pts) Find the average rate of change of  $f(x)$  on the interval  $[1, 1.5]$ .

$$\frac{f(1.5) - f(1)}{1.5 - 1} = \frac{f(3/2) - f(1)}{1/2} = \frac{(2/3) - 1}{1/2} = \frac{-1/3}{1/2} = -\frac{2}{3}$$

(c)(3 pts) Find the average rate of change of  $f(x)$  on the interval  $[1, 1.1]$ .

$$\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{f(11/10) - 1}{1/10} = \frac{(10/11) - 1}{1/10} = \frac{-1/11}{1/10} = -\frac{10}{11}$$

(d)(11 pts) Find  $f'(1)$ .

$$\frac{f(x) - f(1)}{x - 1} = \frac{1/x - 1}{x - 1} = \frac{1/x - x/x}{x - 1} = \frac{1 - x}{x} \times \frac{1}{x - 1} = -\frac{1}{x} \rightarrow -1 \text{ as } x \rightarrow 1$$

Hence  $f'(1) = -1$ .