Worksheet 4 - Solutions

1. Find y' if $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

$$y = x^{1/2} + x^{-4/3} \implies y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \frac{1}{2\sqrt{x}} - \frac{4}{3\sqrt[3]{x^7}}$$

2. Find y' if $y = \tan^2(\sin\theta)$. Using the chain rule,

$$\begin{aligned} y' &= 2\tan(\sin\theta)\frac{d}{d\theta}\tan(\sin\theta) = 2\tan(\sin\theta)\sec^2(\sin\theta)\frac{d}{d\theta}\sin\theta \\ &= 2\tan(\sin\theta)\sec^2(\sin\theta)\cos\theta \end{aligned}$$

3. Find y' if $y = \sqrt[5]{x\tan x}$. Since $y = (x\tan x)^{1/5}$, we use the chain rule:

$$y' = \frac{1}{5}(x\tan x)^{-4/5}\frac{d}{dx}(x\tan x) = \frac{1}{5\sqrt[5]{(x\tan x)^4}}(x\sec^2 + \tan x)$$

4. Find the equation of the tangent line to $y = \sqrt{1 + 4\sin x}$ at the point $(0, 1)$.

$$\begin{aligned} y = (1 + 4\sin(x))^{1/2} &\implies y' = \frac{1}{2}(1 + 4\sin x)^{-1/2}(4\cos x) = \frac{2\cos x}{\sqrt{1 + 4\sin x}} \\ &\implies m = y'(0) = \frac{2\cos 0}{\sqrt{1 + 4\sin 0}} = 0 \\ y - y_1 = m(x - x_1) &\implies y - 1 = 2(x - 0) = 2x \implies y = 2x + 1 \end{aligned}$$

5. At what point on the curve $y = \sin x + \cos x$, $0 \le x \le 2\pi$, is the tangent line horizontal? The cureve is horizontal when $0 = y'$. So

$$0 = y' = \cos x - \sin x \implies \sin x = \cos x \implies x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$y\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \implies (x, y) = \left(\frac{\pi}{4}, \sqrt{2}\right)$$

$$y\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} \implies (x, y) = \left(\frac{5\pi}{4}, -\sqrt{2}\right)$$

6. Find y' if $\sin(xy) = x^2 - y$. Using implicit differentiation, the chain rule, and the product rule

$$\begin{aligned} \frac{d}{dx}\sin(xy) = \frac{d}{dx}(x^2 - y) &\implies \cos(xy)\frac{d}{dx}(xy) = 2x - \frac{dy}{dx} \\ &\implies \cos(xy)\,(xy' + y) = 2x - y' \\ &\implies xy'\cos(xy) + y\cos(xy) = 2x - y' \\ &\implies xy'\cos(xy) + y' = 2x - y\cos(xy) \\ &\implies y'(x\cos xy + 1) = 2x - y\cos(xy) \implies y' = \frac{2x - y\cos(xy)}{x\cos(xy) + 1} \end{aligned}$$

7. Find y' if $x \tan y = y - 1$. Differentiating implicitly,

$$\begin{aligned}\frac{d}{dx}(x \tan y) = \frac{d}{dx}(y-1) &\implies x(\sec^2 y)y' + \tan y = y' \\ &\implies y'(x \sec^2 y - 1) = -\tan y \\ &\implies y' = \frac{\tan y}{1 - x \sec^2 y}\end{aligned}$$

8. Find y'' if $x^6 + y^6 = 1$.

$$\text{Differentiating implicitly: } x^6 + y^6 = 1 \implies 6x^5 + 6y^5 y' = 0 \implies y' = -\frac{x^5}{y^5}$$

$$\text{By the quotient rule: } y'' = -\frac{(y^5)(x^5)' - (x^5)(y^5)'}{y^{10}} = -\frac{5x^4y^5 - 5x^5y^4y'}{y^{10}}$$

$$= -\frac{5x^4y^5 - 5x^5y^4(-x^5/y^5)}{y^{10}} = -\frac{5x^4y^5 + 5x^{10}/y}{y^{10}} = -\frac{5x^4y^6 + 5x^{10}}{y^{11}}$$

9. Find an equation for the tangent line to $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$

$$\text{Differentiate implicitly: } 2x + 4(xy' + y) + 2yy' = 0 \implies 2x + 4xy' + 4y + 2yy' = 0$$

$$\text{Slope } m \text{ is } y' \text{ at } (2,1): 2(2) + 4(2)y' + 4(1) + 2(1)y' = 0 \implies 10m + 8 = 0 \implies m = -4/5$$

$$\text{Eqn of a line: } y - y_1 = m(x - x_1) \implies y - 1 = -\frac{4}{5}(x-2) = -\frac{4}{5}x + \frac{8}{5}$$

$$\implies y = -\frac{4}{5}x + \frac{13}{5}$$

10. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has a slope of 1.

$$\text{Differentiate Implicitly: } 2x + 4yy' = 0$$

$$\text{Solve for } (x, y) \text{ when } y' = 1: 2x + 4y = 0 \implies x = -2y$$

$$\text{Substitute in } x^2 + 2y^2 = 1: (-2y)^2 + 2y^2 = 1 \implies 6y^2 = 1 \implies y = \pm 1/\sqrt{6}$$

$$\text{At } y = 1/\sqrt{6}: x = -2y = -2/\sqrt{6}$$

$$\text{At } y = -1/\sqrt{6}: x = -2y = 2/\sqrt{6}$$

$$\text{The points are: } \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \text{ and } \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$