Worksheet 2 - Solutions

1. Suppose that the position of an object is given by $y = t^2 - 6t + 12$.

   (a) Find the position of the particle at $t = 3$, $t = 3.5$, and $t = 4$.

$$\begin{aligned} y(3) &= 3^2 - 6(3) + 12 = 9 - 18 + 12 = 3 \\ y(3.5) &= (3.5)^2 - 6(3.5) + 12 = 12.25 - 21 + 12 = 3.25 \\ y(4) &= 4^2 - 6(4) + 12 = 16 - 24 + 12 = 4 \end{aligned}$$

   (b) Find the average velocity on [3, 4]

$$v = \frac{y(4) - y(3)}{4 - 3} = \frac{4 - 3}{1} = 1$$

   (c) Find the average velocity on [3.5, 4]

$$v = \frac{y(4) - y(3.5)}{4 - 3.5} = \frac{4 - 3.25}{.5} = \frac{.75}{.5} = 1.5$$

2. Find $f(t + h)$ for the function in problem (1).

$$f(t + h) = (t + h)^2 - 6(t + h) - 12 = t^2 + 2th + h^2 - 6t - 6h + 12$$

3. Find $f(3 + h)$ for the function in problem (1).

$$\begin{aligned} f(3 + h) &= (3)^2 + 2(3)h + h^2 - 6(3) - 6h + 12 \\ &= 9 + 6h + h^2 - 18 - 6h + 12 \\ &= 3 + h^2 \end{aligned}$$

4. Find $\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$ for the function in problem 1.

$$\frac{f(3 + h) - f(3)}{h} = \frac{(3 + h^2) - 3}{h} = \frac{h^2}{h} = h \to 0$$

5. Find $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ for the function in problem 1. The answer will depend on a.

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 - 6a - 6h + 12) - (a^2 - 6a + 12)}{h} \\ &= \frac{2ah + h^2 - 6h}{h} = 2a + h - 6 \to 2a - 6 \end{aligned}$$

6. Let $f(x) = x^2 - 12x$. Then $\lim_{x \to 6} f(x) = -36$. To prove this, find a δ such that

$$|x - 6| < \delta \implies |(x^2 - 12x) - (-36)| < \epsilon$$

   The answer will depend upon ϵ. Then find a value of δ that works for $\epsilon = 0.01$.

$$\begin{aligned} |(x^2 - 12x) - (-36)| < \epsilon &\iff |x^2 - 12x + 36| < \epsilon \\ &\iff |(x - 6)^2| < \epsilon \\ &\iff |x - 6|^2 < \epsilon \\ &\iff |x - 6| < \sqrt{\epsilon} \end{aligned}$$

   Choose $\delta = \sqrt{\epsilon}$. For $\epsilon = 0.01$ this means $\delta = \sqrt{.01} = .1$.

7. Find what value of the constant c will $f(x) = \begin{cases} x^2 - c, & \text{if } x < 7 \\ cx + 7, & \text{if } x \geq 7 \end{cases}$ be continuous on $(-\infty, \infty)$.

$$\begin{aligned} \text{at } x = 7 \text{ we require } x^2 - c &= cx + 7 \\ \implies 7^2 - c &= 7c + 7 \\ \implies 49 - c &= 7c + 7 \\ \implies 42 &= 8c \\ \implies c &= \frac{42}{8} = 5.25 \end{aligned}$$

8. Let $f(x) = 1/x^2$. Find $f'(5)$ using the definition of the derivative.

$$\begin{aligned} f(5) &= \frac{1}{25} \\ f(5+h) &= \frac{1}{(5+h)^2} \\ f(5+h) - f(5) &= \frac{1}{(5+h)^2} - \frac{1}{25} = \frac{25 - (5+h)^2}{25(5+h)^2} = \frac{25 - 25 - 10h - h^2}{25(5+h)^2} = -\frac{h(10+h)}{25(5+h)^2} \\ \frac{f(5+h) - f(5)}{h} &= -\frac{10+h}{25(5+h)^2} \\ \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} &= \lim_{h \to 0} -\frac{10+h}{25(5+h)^2} = -\frac{10}{(25)(25)} = -\frac{2}{125} \end{aligned}$$

9. Let $f(x) = -2x^3 + 6x + 1$. Find $f'(x)$ and $f''(x)$ as a function of x using the defintion of a derivative.

$$\begin{aligned} f(x+h) - f(x) &= \big(-2(x+h)^3 + 6(x+h) + 1\big) - (-2x^3 + 6x + 1) \\ &= -2(x^3 + 3x^2h + 3xh^2 + h^3) + 6x + 6h + 1 + 2x^3 - 6x - 1 \\ &= -2x^3 - 6x^2h - 6xh^2 - 2h^3 + 6h + 2x^3 \\ &= -6x^2h - 6xh^2 - 2h^3 + 6h \\ \frac{f(x+h) - f(x)}{h} &= \frac{-6x^2h - 6xh^2 - 2h^3 + 6h}{h} = -6x^2 - 6xh - 2h + 6 \\ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= -6x^2 + 6 \\ f'(x+h) - f'(x) &= \big(-6(x+h)^2 + 6\big) - (-6x^2 + 6) \\ &= -6(x^2 + 2xh + h^2) + 6 + 6x^2 - 6 \\ &= -12xh - 6h^2 \\ f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} &= \lim_{h \to 0} \frac{-12xh - 6h^2}{h} = \lim_{h \to 0}(-12x - 6h) = -12x \end{aligned}$$