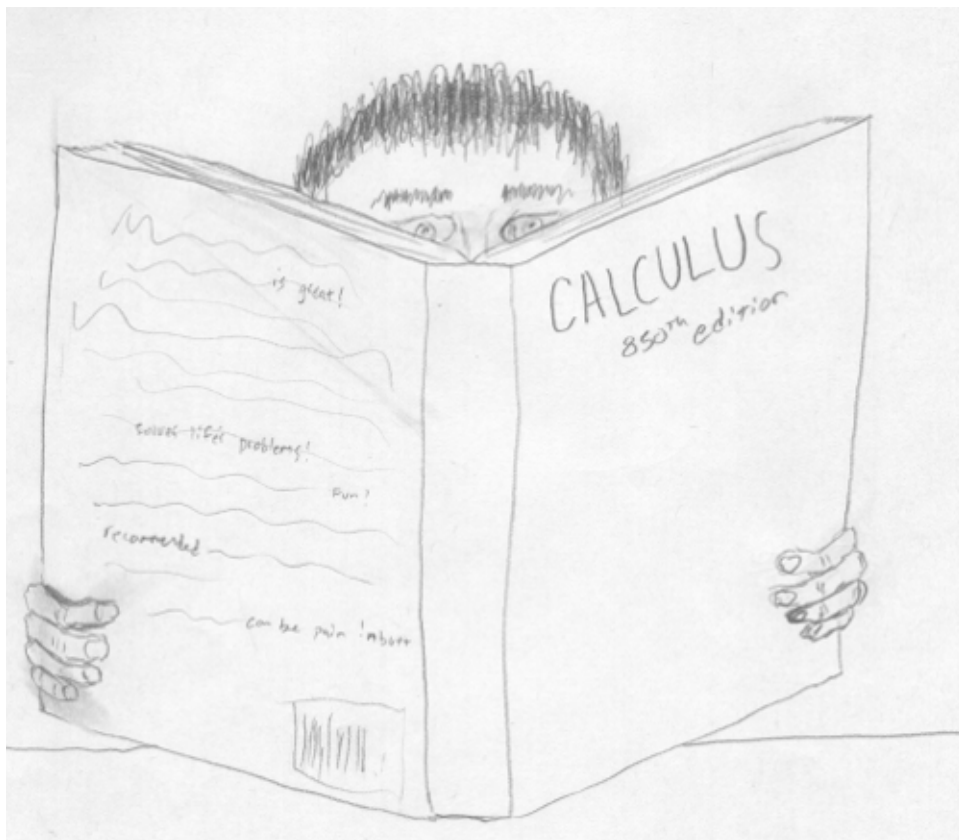


Math 150A, Fall 2009
Lecture Notes

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Last Revised: December 5, 2009

Note to the Student

Don't try to use this as a replacement for your textbook. These are notes and just provide an outline of the subject material, not a complete presentation. I have provided a copy to you to use only as a study aid. Their real purpose is to remind me what to talk about during my class lectures. You should read the textbook in preparation for class, and just use these notes to aide your own note-taking during class.

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Please report any errors to bruce.e.shapiro@csun.edu. All feedback, comments, suggestions for improvement, etc., is appreciated, especially if you've used these notes for a class, either at CSUN or elsewhere, from both instructors and students.

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Topic 1

Limits

Example 1.1 Find an equation of the tangent line to $y = x^2$ at the point $P(1, 1)$.

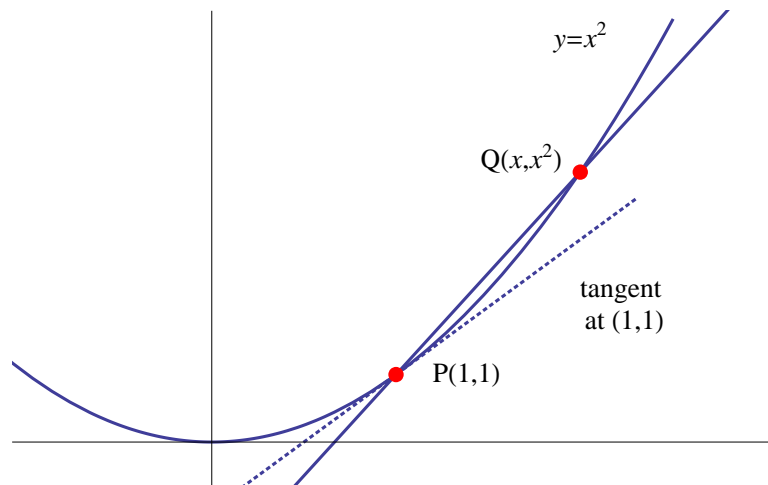
Method: Choose a point (x, y) near $(1, 1)$, with $x \neq 1$. Then $Q = (x', y') = (x, x^2)$ so that

$$m = \frac{\text{rise}}{\text{run}} = \frac{y' - y}{x' - x} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad (1.1)$$

We can approximate values of the slope by taking values for x as x gets closer and closer to 1:

x	m
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.00001	2.00001
\vdots	\vdots
1	2

x	m
0	1
0.5	1.5
.9	1.9
0.99	1.99
0.99999	1.99999
\vdots	\vdots
1	2



Whether we come from the left or from the right, we “approach” the same “limiting value.”

We conclude then that at $x = 1$, the slope is $m = 2$.

To find the equation of this line we can write $y = mx + b$ with $(x, y) = (1, 1)$ and $m = 2$ to give $1 = (2)(1) + b \implies b = -1$, hence

$$y = 2x - 1 \tag{1.2}$$

is the equation of tangent line to the parabola through $(1, 1)$.

Definition 1.1 Average Rate of Change. The average rate of change of $f(x)$ between $x = a$ and $x = b$ is defined as

$$\text{average rate of change on } [a, b] = \frac{f(b) - f(a)}{b - a} \tag{1.3}$$

If we write $b - a = h$ then

$$\text{average rate of change on } [a, a + h] = \frac{f(a + h) - f(a)}{h} \tag{1.4}$$

Definition 1.2 Instantaneous Rate of Change The instantaneous rate of change of $f(x)$ at $x = a$ is the slope of the tangent line to $f(x)$ at $x = a$. It is the limiting value of the average rate of change on $[a, b]$ as b becomes very close to a .

Remark 1.3 The instantaneous rate of change of $f(x)$ at $x = a$ is only defined when a unique tangent line exists at $x = a$.

If there is a jump in the function then the instantaneous rate of change is undefined.

Definition 1.4 Velocity Problem According to Galileo's Law if an object is dropped then (neglecting air resistance), the distance $s(t)$ it has fallen in meters, after a time t , is

$$s(t) = 4.9t^2 \quad (1.5)$$

Example 1.2 Suppose a ball is dropped from a height of 450 meters. What is its velocity after 5 seconds?

We can use the limiting process to estimate the balls average speed, say between $t = 5$ and $t = x$ and then slowly reduce x to zero estimate this speed.

We calculate

$$\text{average rate of speed between 5 and } x = \frac{s(5) - s(x)}{5 - x} \quad (1.6)$$

$$= \frac{4.9(5^2) - 4.9x^2}{5 - x} \quad (1.7)$$

$$= 4.9 \frac{25 - x^2}{5 - x} \quad (1.8)$$

$$= 4.9 \frac{(5 - x)(5 + x)}{5 - x} \quad (1.9)$$

$$= 4.9(5 + x) \quad (1.10)$$

$$= 24.5 + 4.9x \quad (1.11)$$

As $x \rightarrow 5$ we obtain

$$\text{speed} \rightarrow 24.5 + (4.9)(5) = 49.0 \quad (1.12)$$

Definition 1.5 Limit. We say that the limit of $f(x)$, as x approaches a , equals L , and write

$$\lim_{x \rightarrow a} f(x) = L \quad (1.13)$$

or

$$f(x) \rightarrow L \text{ as } x \rightarrow a \quad (1.14)$$

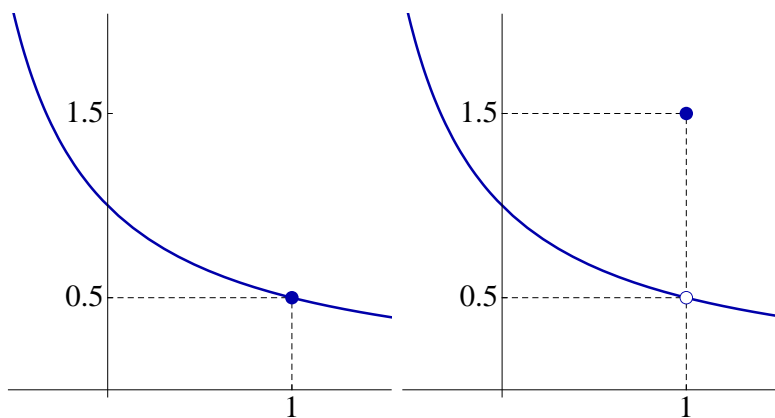
if we can make the value of $f(x)$ as close to L as we like, by choosing x sufficiently close to a (but not equal to a), approaching a from both sides (the left and the right).

Example 1.3 Show that

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5 \quad (1.15)$$

Example 1.4 Show that $\lim_{x \rightarrow 1} f(x)$ does not exist for

$$f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1.5 \\ 3 & \text{if } x = 1.5 \end{cases} \quad (1.16)$$



Example 1.5 Use the calculator to guess

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (1.17)$$

x	$f(x)$
1	0.8414709848078965
0.1	0.9983341664682815
0.01	0.9999833334166665
0.001	0.999998333333416
0.0001	0.99999983333334
\vdots	
$x \rightarrow 0$	1

We will see later that this guessing method gives the correct answer.

Remark 1.6 Guessing the value of a limit can lead to numerical errors, if you use a calculator.

Example 1.6 Try $x = .0000000001$ in you calculator with the following:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{1}{6} \quad (1.18)$$

Your calculator will probably detrmine that

$$\sqrt{.0000000001^2 + 9} = 3 \quad (1.19)$$

and hence

$$\frac{\sqrt{.0000000001^2 + 9} - 3}{.0000000001^2} = \frac{0}{.0000000001^2} = 0 \quad (1.20)$$

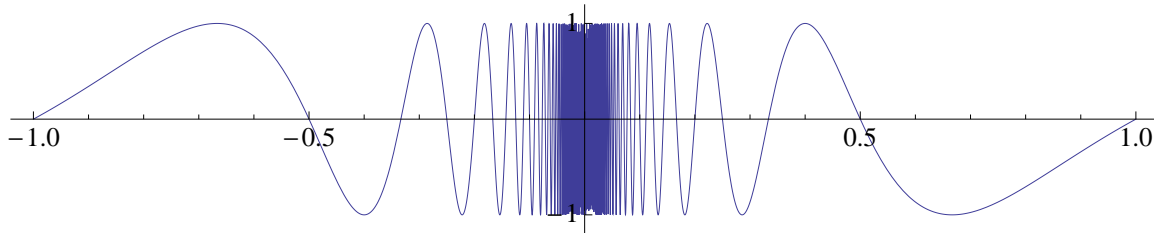
Example 1.7 Estimate

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} \tag{1.21}$$

If we just blindly plug in numbers we might get something like this:

x	$f(x)$
1	$\sin \pi = 0$
.5	$\sin 2\pi = 0$
.1	$\sin 10\pi = 0$
.001	$\sin 1000\pi = 0$
\vdots	
$\lim_{x \rightarrow 0}$	0

But we would be wrong, as the following figure illustrates. In fact, this limit does not exist, because no matter how close we get to 0, the function takes on infinitely many values between -1 and 1.



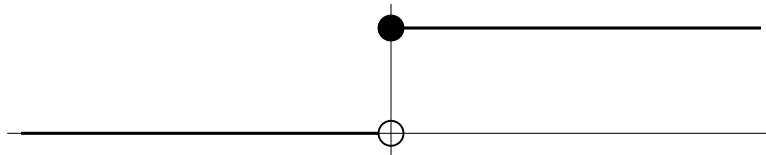
Topic 2

Calculating Limits

One-Sided Limits

Definition 2.1 Heaviside (Step) Function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad (2.1)$$



If we try calculating

$$\lim_{x \rightarrow 0} H(x) \quad (2.2)$$

we find that

$$\text{as } x \rightarrow 0 \text{ from the left, } H(x) \rightarrow 0 \quad (2.3)$$

$$\text{as } x \rightarrow 0 \text{ from the right, } H(x) \rightarrow 1 \quad (2.4)$$

Since we get different values approaching from the left and the right then the quantity

$$\lim_{x \rightarrow 0} H(x) \quad (2.5)$$

does not exist.

Instead, we write

$$\lim_{x \rightarrow 0^-} H(x) = 0, \quad \lim_{x \rightarrow 0^+} H(x) = 1 \quad (2.6)$$

or

$$H(x) \rightarrow 0 \text{ as } x \rightarrow 0^- \quad (2.7)$$

$$H(x) \rightarrow 1 \text{ as } x \rightarrow 0^+ \quad (2.8)$$

The minus sign ($-$) in the superscript is read as “from the left.”

The plus sign ($+$) in the superscript is read as “from the right.”

Definition 2.2 We say “the left hand limit of $f(x)$ as x approaches a equals L ” or “the limit of $f(x)$ from the left as x approaches a equals L ”, and write

$$\lim_{x \rightarrow a^-} f(x) = L \quad (2.9)$$

if we can make the value of $f(x)$ as close to L as we like by choose x as close to a (but never equal to a) as we like *on the **left** hand side* ($x < a$).

We say “the right hand limit of $f(x)$ as x approaches a equals L ” or “the limit of

$f(x)$ from the right as x approaches a equals L ", and write

$$\lim_{x \rightarrow a^+} f(x) = L \quad (2.10)$$

if we can make the value of $f(x)$ as close to L as we like by choose x as close to a (but never equal to a) as we like *on the **right** hand side* ($x > a$)).

Remark 2.3 The value of $f(a)$ is irrelevant in computing any of the limits as $x \rightarrow a$, $x \rightarrow a^-$, or $x \rightarrow a^+$. It does not even have to be defined.

Remark 2.4 The symbol \iff is read as "if and only if," and we write $A \iff B$.

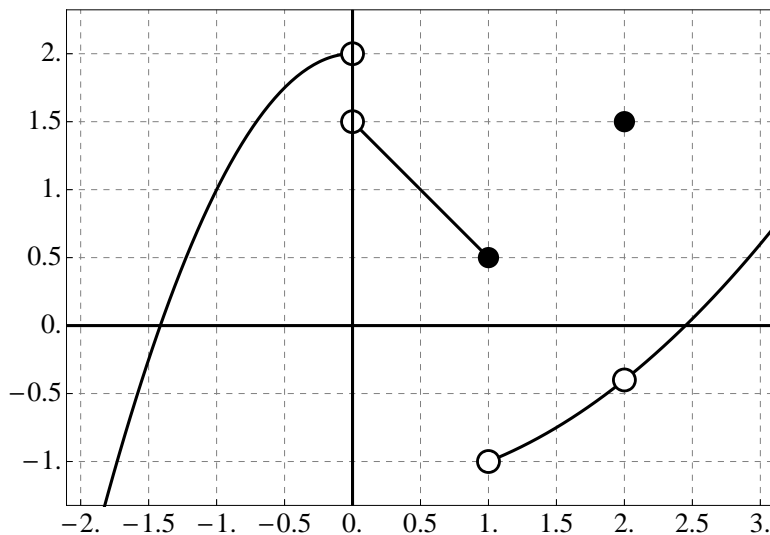
This means two things:

(1) A implies B, which we write as $A \implies B$; and

(2) B implies A, which we write as $B \implies A$.

Definition 2.5 $\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L$

Example 2.1 In the following figure,



find the following limits:

$$\begin{array}{lll}
 \text{(a)} \lim_{x \rightarrow 0^-} f(x) & \text{(e)} \lim_{x \rightarrow 1^+} f(x) & \text{(i)} \lim_{x \rightarrow 2} f(x) \\
 \text{(b)} \lim_{x \rightarrow 0^+} f(x) & \text{(f)} \lim_{x \rightarrow 1} f(x) & \text{(j)} \lim_{x \rightarrow -1^-} f(x) \\
 \text{(c)} \lim_{x \rightarrow 0} f(x) & \text{(g)} \lim_{x \rightarrow 2^-} f(x) & \text{(k)} \lim_{x \rightarrow -1^+} f(x) \\
 \text{(d)} \lim_{x \rightarrow 1^-} f(x) & \text{(h)} \lim_{x \rightarrow 2^+} f(x) & \text{(l)} \lim_{x \rightarrow -1} f(x)
 \end{array}$$

Infinite Limits and Vertical Asymptotes

Definition 2.6 Suppose $f(x)$ is a function that is defined for all x in some interval that includes a , except possibly at a itself. Then we say that the limit as x approaches a of $f(x)$ is **infinity**, or **positive infinity** which we write as,

$$\lim_{x \rightarrow a} f(x) = \infty \quad (2.11)$$

to mean that the values of $f(x)$ can be made as large as we like by taking x sufficiently close to (but not equal to) a , and we say that the limit as x approaches a of $f(x)$ is **negative infinity**, which we write as,

$$\lim_{x \rightarrow a} f(x) = -\infty \quad (2.12)$$

to mean that the values of $f(x)$ can be made as large negative as we like by taking x sufficiently close to (but not equal to) a .

Definition 2.7 The line $x = a$ is called a **vertical asymptote** of $f(x)$ if any of the following are true:

$$\left. \begin{array}{lll}
 \lim_{x \rightarrow a} f(x) = \infty & \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\
 \lim_{x \rightarrow a} f(x) = -\infty & \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty
 \end{array} \right\} \quad (2.13)$$

Example 2.2 Find

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3}, \quad \lim_{x \rightarrow 3^+} \frac{2x}{x-3} \quad (2.14)$$

Limit Laws

Theorem 2.8 If the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

1. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
2. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$ (where c is any constant)
3. $\lim_{x \rightarrow a} [f(x)g(x)] = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right)$
4. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ (so long as $\lim_{x \rightarrow a} g(x) \neq 0$)
5. $\lim_{x \rightarrow a} c = c$
6. $\lim_{x \rightarrow a} x = a$
7. $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
8. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$
9. **Direct Substitution.** If $f(x)$ is a rational function or polynomial and x is in the domain of $f(x)$ then $\lim_{x \rightarrow a} f(x) = f(a)$.

Example 2.3

$$\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$$

Example 2.4

$$\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Example 2.5

$$\lim_{x \rightarrow 5} \frac{x^2 - 5}{x - 5}$$

Example 2.6 Rationalized the numerator to find

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

Example 2.7 Define

$$f(x) = \begin{cases} 8 - 2x & \text{if } x < 4 \\ \sqrt{x - 4} & \text{if } x > 4 \end{cases} \quad (2.15)$$

Does $\lim_{x \rightarrow 4}$ exist, and if so, what is its value?

Theorem 2.9 Squeeze Theorem (or the **Sandwich Theorem** or the **Pinching Theorem**). If $f(x) \leq g(x) \leq h(x)$ when x is near a and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x) \quad (2.16)$$

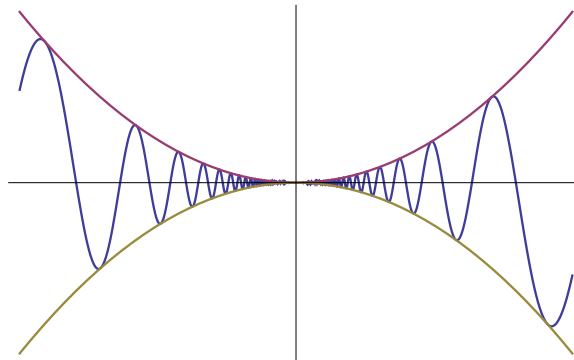
then

$$\lim_{x \rightarrow a} g(x) = L \quad (2.17)$$

Example 2.8 Show that

$$\lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0 \quad (2.18)$$

by pinching it between $f(x) = -x^2$ and $h(x) = x^2$.



Topic 3

Limits: Theory

Lets recall the definition of

$$\lim_{x \rightarrow a} f(x) = L \quad (3.1)$$

which means we can make the value of $f(x)$ as close to L as we like, by choosing x sufficiently close (but not equal) to a (but not equal to a).

Lets try and figure out what this means more precisely from an example. We will start by examining

$$\lim_{x \rightarrow 4} \sqrt{x} = L \quad (3.2)$$

We know from theorem 2.8 law 9 (Direct Substitution) that $L = 2$.

What does this mean, more precisely, from the definition?

It means that we can make $f(x)$ as close as we like to L by choosing x sufficiently close to a , where

$$f(x) = \sqrt{x}, \quad L = 2, \quad a = 4 \quad (3.3)$$

What do we mean by “ $f(x)$ as close as we like to L ?” It means we pick some small

number, say 0.00001, and then we must have

$$|f(x) - L| < \text{whatever small number we choose, say } 0.00001 \quad (3.4)$$

or, more precisely, for our example,

$$|\sqrt{x} - 2| < 0.00001 \quad (3.5)$$

The limit definition says that for some x , very close to a , equation 3.5 is satisfied.

How close must we be? We'll call this distance δ . Then the limit definition says that for some δ , whenever

$$|x - a| \text{ is sufficiently small, i.e., } < \delta \text{ for some small } \delta \quad (3.6)$$

or more precisely, whenever

$$|x - 4| < \delta \quad (3.7)$$

then

$$|\sqrt{x} - 2| < 0.00001 = 10^{-5} \quad (3.8)$$

Let's see if we can figure out what value δ must be to make this work. We need to ultimately get:

$$-10^{-5} < \sqrt{x} - 2 < 10^{-5} \quad (3.9)$$

starting from

$$-\delta < x - 4 < \delta \quad (3.10)$$

$$-\delta < (\sqrt{x} - 2)(\sqrt{x} + 2) < \delta \quad (3.11)$$

$$-\frac{\delta}{\sqrt{x}+2} < \sqrt{x}-2 < \frac{\delta}{\sqrt{x}+2} \quad (3.12)$$

Observe now that equation 3.12 looks something like equation 3.9 with 10^{-5} replaced by

$$\frac{\delta}{\sqrt{x}+2} \quad (3.13)$$

What we really want is for 3.12 to *imply* equation 3.9. One way to do this is by finding a δ that satisfies:

$$-10^{-5} < -\frac{\delta}{\sqrt{x}+2} < \sqrt{x}-2 < \frac{\delta}{\sqrt{x}+2} < 10^{-5} \quad (3.14)$$

and one way to do this is to find a δ that satisfies

$$\delta < 10^{-5}(\sqrt{x}+2) \quad (3.15)$$

Now, going back to the original problem of finding $\lim_{x \rightarrow 4} \sqrt{x}$, we want x to be sufficiently close to 4. So that means that

$$\sqrt{x}+2 \approx \sqrt{4}+2 = 4 \quad (3.16)$$

so we really want is to get

$$\text{approximately } \left(\frac{\delta}{4}\right) < 10^{-5} \quad (3.17)$$

say,

$$\frac{\delta}{4} < \frac{10^{-5}}{4} < 10^{-5} \quad (3.18)$$

We could pick

$$\delta \approx 10^{-5} \tag{3.19}$$

We could also pick $\delta \approx 10^{-6}$ or $\delta \approx .3 \times 10^{-7}$, etc., so long as we satisfy $\delta/4 < 10^{-5}$.

So lets work our way backwards now. Lets suppose that

$$\delta = 10^{-5} \tag{3.20}$$

The consequence is that

$$|x - 4| < 10^{-5} \implies -10^{-5} < x - 4 < 10^{-5} \tag{3.21}$$

$$\implies -10^{-5} < (\sqrt{x} - 2)(\sqrt{x} + 2) < 10^{-5} \tag{3.22}$$

$$\implies -\frac{10^{-5}}{\sqrt{x} + 2} < \sqrt{x} - 2 < \frac{10^{-5}}{\sqrt{x} + 2} \tag{3.23}$$

Now since $x - 4 < 10^{-5}$ then $x < 4 + 10^{-5}$ so

$$\sqrt{x} + 2 < \sqrt{4 + 10^{-5}} + 2 > 4 \implies \frac{1}{\sqrt{x} + 2} < \frac{1}{4} \tag{3.24}$$

and, if we multiply by -1 ,

$$-\frac{1}{4} < -\frac{1}{\sqrt{x} + 2} \tag{3.25}$$

So if we plug equations 3.24 and 3.25 into equation 3.23 we get

$$-\frac{10^{-5}}{4} < \sqrt{x} - 2 < \frac{10^{-5}}{4} \tag{3.26}$$

Now since $1/4 < 1$,

$$-10^{-5} < -\frac{10^{-5}}{4} < \sqrt{x} - 2 < \frac{10^{-5}}{4} < 10^{-5} \tag{3.27}$$

and finally,

$$|\sqrt{x} - 2| < 10^{-5} \tag{3.28}$$

So if we look back over our steps starting with equation 3.20 and going through equation 3.28, we have shown how

$$|x - 4| < \delta = 10^{-5} \implies |\sqrt{x} - 2| < 10^{-5} \tag{3.29}$$

or in terms of the original limit problem ($a = 4$, $L = 2$, $f(x) = \sqrt{x}$), given $\epsilon = 10^{-5}$, we found that $\delta = 10^{-5}$ is sufficient to prove that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon \tag{3.30}$$

This is precisely the theoretical definition of a limit.

Definition 3.1 Limit, Precise Definition. Let $f(x)$ be a function that is defined on some open interval containing a , except possibly at the number a itself (this means that f does not have to be defined at a , although it's allowed to be defined there).

Then we say *the limit of $f(x)$ as x approaches a is L* if:

Given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|x - a| < \delta \implies |f(x) - L| < \epsilon \tag{3.31}$$

and we write

$$\lim_{x \rightarrow a} f(x) = L \tag{3.32}$$

Less precisely, perhaps, we can state it this way. $\lim_{x \rightarrow a} f(x) = L$ means the following: I let you pick any number ϵ , as small as you like. (When we use letters like ϵ and δ

we always mean very small numbers.) Then I can find another number δ , which is also likely to be very small, and possibly equal to ϵ , such that whenever x is within δ (my number) of a , then $f(x)$ is within ϵ (your number) of L .

The left and right limits are also defined precisely in a similar manner, but without the need for absolute values on the δ equation.

Definition 3.2 Left Hand Limit, Precise Definition. $\lim_{x \rightarrow a^-} f(x) = L$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \epsilon \quad (3.33)$$

Definition 3.3 Right Hand Limit, Precise Definition. $\lim_{x \rightarrow a^+} f(x) = L$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \epsilon \quad (3.34)$$

Example 3.1 Prove that $\lim_{x \rightarrow 5} x^3 = 125$

Let ϵ be given. Doing the analysis as before, we want to find δ such that $|x - 5| < \delta \implies |x^3 - 125| < \epsilon$. So working backwards from the ϵ equation, we want

$$|x^3 - 125| < \epsilon \implies -\epsilon < x^3 - 125 < \epsilon \quad (3.35)$$

$$\implies -\epsilon < (x - 5)(x^2 + 5x + 25) < \epsilon \quad (3.36)$$

$$\implies -\frac{\epsilon}{x^2 + 5x + 25} < x - 5 < \frac{\epsilon}{x^2 + 5x + 25} \quad (3.37)$$

So we want to pick a δ that is somewhat smaller than

$$\delta < \frac{\epsilon}{x^2 + 5x + 25} \approx \frac{\epsilon}{75} \text{ for } x \text{ near } 5 \quad (3.38)$$

So we can arbitrarily pick, say, $\delta = \epsilon/100$, because $\epsilon/100 < \epsilon/75$. We could also pick any other number for δ so long as it is smaller than $\epsilon/75$.

Then the proof goes like this: Let $\epsilon > 0$ be given. Pick $\delta = \epsilon/100$. Then

$$|x - 5| < \delta = \frac{\epsilon}{100} \implies -\frac{\epsilon}{100} < x - 5 < \frac{\epsilon}{100} \quad (3.39)$$

But since $x - 5 < \epsilon/100$ then $x < 5 + \epsilon/100$ which means

$$x^2 + 5x + 25 < \left(x + \frac{\epsilon}{100}\right)^2 + 5\left(x + \frac{\epsilon}{100}\right) + 25 \quad (3.40)$$

$$< 75 + \text{a very small number} \quad (3.41)$$

$$< 100 \quad (3.42)$$

$$\frac{1}{100} < \frac{1}{x^2 + 5x + 25} \quad (3.43)$$

$$-\frac{1}{x^2 + 5x + 25} < -\frac{1}{100} \quad (3.44)$$

hence substituting the last two equations into equation 3.39

$$-\frac{\epsilon}{x^2 + 5x + 25} < x - 5 < \frac{\epsilon}{x^2 + 5x + 25} \quad (3.45)$$

$$\implies -\epsilon < (x - 5)(x^2 + 5x + 25) < \epsilon \quad (3.46)$$

$$\implies -\epsilon < x^3 - 125 < \epsilon \quad (3.47)$$

$$\implies |x^3 - 125| < \epsilon \quad (3.48)$$

Thus for any $\epsilon > 0$ (you choose), I can find a $\delta > 0$ (by setting $\delta = \epsilon/100$) such that

whenever $|x - 5| < \delta$ then $|x^3 - 125| < \epsilon$. This is the strict definition of $\lim_{x \rightarrow 5} x^3 = 125$

Definition 3.4 Infinite Limit, Precise Definition. Let f be defined on an open interval that contains a , except possibly at a (meaning: f does not have to be defined at a , although it is allowed to be defined at a). Then

$$\lim_{x \rightarrow a} f(x) = \infty \quad (3.49)$$

means that for every $M > 0$ (no matter how large) there is a positive number $\delta > 0$ (that is usually very small) such that

$$|x - a| < \delta \implies f(x) > M \quad (3.50)$$

and

$$\lim_{x \rightarrow a} f(x) = -\infty \quad (3.51)$$

means that for every $N < 0$ (no matter how large and negative) there is a positive number $\delta > 0$ (that is usually very small) such that

$$|x - a| < \delta \implies f(x) < N \quad (3.52)$$

Example 3.2 Prove that $\lim_{x \rightarrow 0} (1/x^2) = \infty$.

Let $M > 0$ be given. We need to find $\delta > 0$ such that $|x - 0| < \delta \implies 1/x^2 > M$.

We observe that

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}} \quad (3.53)$$

Reversing the steps in the last equation, if we choose $\delta = 1/\sqrt{M}$ then $|x| < \delta \implies 1/x^2 > M$, as desired.

Topic 4

Continuity

Definition 4.1 Continuity at a Point. A function $f(x)$ is said to be **continuous at the number a** if

$$\lim_{x \rightarrow a} f(x) = f(a) \tag{4.1}$$

If $f(x)$ is not continuous at a we say that $f(x)$ is **discontinuous at a** .

This definition really says that three things must be true for a function to be continuous at $x = a$:

1. $f(a)$ must be defined, i.e., $f(x)$ is defined at $x = a$; and
2. $\lim_{x \rightarrow a} f(x)$ must exist; and
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

In general, functions are discontinuous at the following places:

1. Where there is a “hole” in the function; and
2. Where there is a “jumps” in the function; and
3. Where the function goes to $\pm\infty$ or has a vertical asymptote.

and generally, the function is continuous every place else.

One way of describing continuity is this: try to draw a graph of the function using a pencil. Any place that you can draw the graph without lifting the pencil, the function is continuous; but if you have to lift up the pencil (or erase a point), the function is discontinuous at those places where you must lift the pencil.

Definition 4.2 Directional Continuity. We say that $f(x)$ is **continuous from the right at a** if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad (4.2)$$

and **continuous from the left at a** if

$$\lim_{x \rightarrow a^-} f(x) = f(a) \quad (4.3)$$

Definition 4.3 A function is **continuous on an interval (a, b)** if it is continuous at every point in the interval.

Definition 4.4 The **interval of continuity** is the set of all points in the domain of a function $f(x)$ at which the function is continuous.

Remark 4.5 A function is only continuous at points where it is defined. Thus, for example, \sqrt{x} is not continuous at any point for which $x < 0$; is continuous from the right at $x = 0$; and is continuous on $(0, \infty)$.

Example 4.1 Determine where the function

$$f(x) = 1 - \sqrt{2 - x^2} \quad (4.4)$$

is continuous.

First of all, we need to determine where $f(x)$ is defined (i.e., find its domain). The function has a square root, which is only defined for non-negative argument; hence we require

$$2 - x^2 \geq 0 \implies x^2 \leq 2 \implies -\sqrt{2} \leq x \leq \sqrt{2} \quad (4.5)$$

so the domain of $f(x)$ is $[-\sqrt{2}, \sqrt{2}]$.

To see where the function is continuous, we need to consider (a) the interior of the interval $(-\sqrt{2}, \sqrt{2})$; and (b) the endpoints $x = \pm\sqrt{2}$.

In the interior of the interval, we can apply the limit laws:

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (1 - \sqrt{2 - x^2}) \quad (4.6)$$

$$= \lim_{x \rightarrow a} 1 - \lim_{x \rightarrow a} \sqrt{2 - x^2} \quad (\text{limit of a sum}) \quad (4.7)$$

$$= 1 - \lim_{x \rightarrow a} \sqrt{2 - x^2} \quad (\text{limit of a constant}) \quad (4.8)$$

$$= 1 - \sqrt{\lim_{x \rightarrow a} (2 - x^2)} \quad (\text{limit of a square root}) \quad (4.9)$$

$$= 1 - \sqrt{2 - a^2} \quad (\text{substitution law for polynomials}) \quad (4.10)$$

$$= f(a) \quad (\text{definition of } f(x)) \quad (4.11)$$

The same sequence of steps applies at the left endpoint using $x \rightarrow (-\sqrt{2})^+$ and the right endpoint using $x \rightarrow (\sqrt{2})^-$. Hence the function is continuous on the entire interval $[-\sqrt{2}, \sqrt{2}]$.

Theorem 4.6 Continuity Laws. If $f(x)$ and $g(x)$ are continuous at a then

1. $f(x) + g(x)$ is continuous at $x = a$;
2. $f(x) - g(x)$ is continuous at $x = a$;
3. $f(x)g(x)$ is continuous at $x = a$;

4. $cf(x)$ is continuous at $x = a$ for any constant c ;
5. $f(x)/g(x)$ is continuous at a if $g(a) \neq 0$

In addition,

6. All polynomials are continuous everywhere (i.e., on $(-\infty, \infty)$);
7. All rational functions are continuous for every point in their domain;
8. All roots are continuous at every point in their domain;
9. All trigonometric functions are continuous at every point in their domain.

Example 4.2 Determine where the following function is continuous:

$$f(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1} \quad (4.12)$$

The first term is a root which is continuous at every point its domain, which is $[0, \infty)$.

The second term is a rational function which is continuous at every point in its domain, which is $(-\infty, 1) \cup (1, \infty)$.

The third term is a rational function which is continuous at every point in its domain, which is $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the intersection of those three sets, which is

$$[0, 1) \cup (1, \infty) \quad (4.13)$$

Theorem 4.7

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) \quad (4.14)$$

Theorem 4.8 If f is continuous at a and f is continuous at $g(a)$ then its composition

$$(f \circ g)(x) = f(g(x)) \quad (4.15)$$

is continuous at $x = a$.

Thus, for example,

$$\lim_{x \rightarrow \sqrt{\pi}} \sin(x^2) = \sin\left(\lim_{x \rightarrow \sqrt{\pi}} x^2\right) = \sin \pi = 0 \quad (4.16)$$

Theorem 4.9 Intermediate Value Theorem. Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ such that $f(a) \neq f(b)$. Let y be any number between $f(a)$ and $f(b)$. Then there exists some number $c \in (a, b)$ such that $f(c) = y$.

In other words, in getting from a to b , f must pass through every number between $f(a)$ and $f(b)$.

Example 4.3 Show that the function

$$f(x) = 4x^3 - 6x^2 + 3x - 2 \quad (4.17)$$

has a root in the interval $[1, 2]$.

We calculate

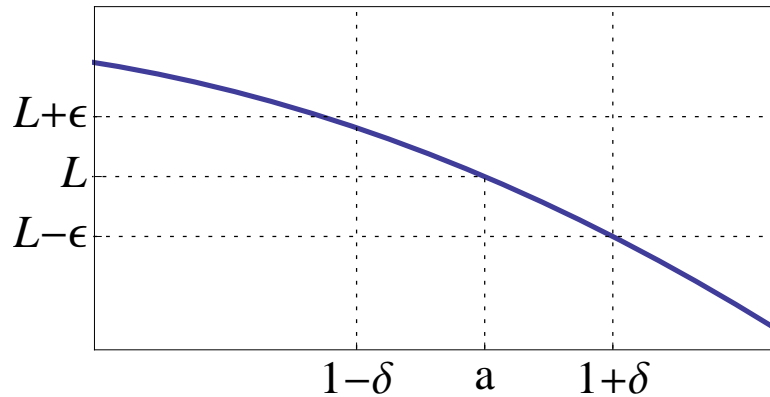
$$f(1) = 4 - 6 + 3 - 2 = -1 \quad (4.18)$$

$$f(2) = 4(8) - 6(4) + 3(2) - 2 = 12 \quad (4.19)$$

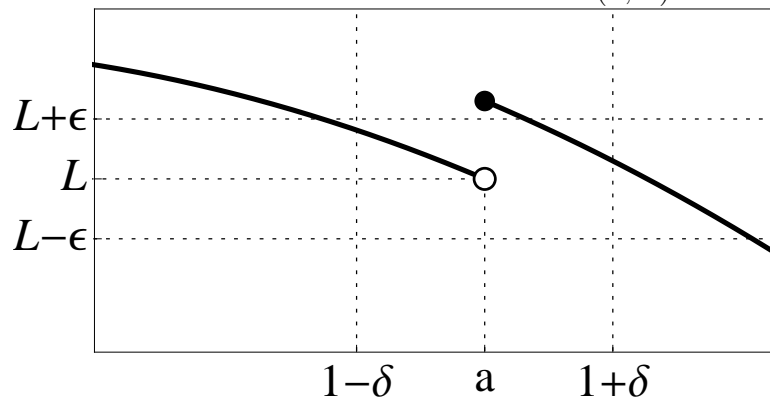
Since $f(1) = -1 < 0$ and $f(2) = 12 > 0$, then there must be some point c in $(1, 2)$

such that $f(c) = 0$. There is a root at c .

Example of a continuous function at $x = a$. No matter how small you choose ϵ I can always find a δ so that the function is within the $2\epsilon \times 2\delta$ box centered at (a, L) .



Example of a function that is discontinuous at $x = a$. For the chosen ϵ shown (and for many ϵ 's that are smaller than this), no matter how small I choose δ it is not possible to find a value that is sufficiently small so that the function is entirely within the $2\epsilon \times 2\delta$ box centered at (a, L) .



Topic 5

The Derivative at a Point

Definition 5.1 The **tangent line** to the curve $y = f(x)$ at the point

$$P(a, f(a)) \tag{5.1}$$

is the line through P with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \tag{5.2}$$

provided that that limit exists.

Example 5.1 Find the equation of the tangent line to $y = 2x^2 + 1$ at the point $P(1, 3)$

At the point $a = 1$, $f(1) = 2(1)^2 + 1 = 3$ and the slope is

$$m = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{2x^2 + 1 - 3}{x - 1} \tag{5.3}$$

$$= \lim_{x \rightarrow 1} \frac{2x^2 - 2}{x - 1} \tag{5.4}$$

Since $2x^2 - 2 = 2(x^2 - 1) = 2(x - 1)(x + 1)$,

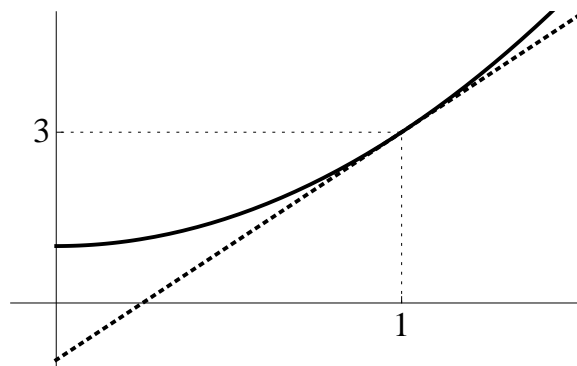
$$m = \lim_{x \rightarrow 1} \frac{2(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1} (2(x + 1)) = 4 \quad (5.5)$$

Hence the tangent line goes through the point (1,3) and has a slope of 4.

$$y = mx + b \implies 3 = (4)(1) + b \quad (5.6)$$

$$\implies b = 3 - 4 = -1 \quad (5.7)$$

$$\implies y = 4x - 1 \quad (5.8)$$



If we substitute

$$h = x - a \quad (5.9)$$

into equation 5.2 and observe that

$$\lim_{x \rightarrow a} h = \lim_{x \rightarrow a} (x - a) = a - a = 0 \quad (5.10)$$

we get a second formula for the slope:

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (5.11)$$

Example 5.2 Use the second definition of the tangent to find the equation of the tangent line to

$$y = \frac{2}{x+1} \quad (5.12)$$

at $x = 1$

Since $f(1) = 2/(1+1) = 2/2 = 1$ we want to find the tangent line through the point $P(1, 1)$. The slope is given by

$$m = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \quad (5.13)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{2}{1+h+1} - 1}{h} \quad (5.14)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{2}{2+h} - \frac{2+h}{2+h}}{h} \quad (5.15)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h(2+h)} \quad (5.16)$$

$$= \lim_{h \rightarrow 0} \frac{-1}{2+h} \quad (5.17)$$

$$= -\frac{1}{2} \quad (5.18)$$

The line through $(1,1)$ with slope $-1/2$ has equation $y = -x/2 + b$; substituting $(1,1)$ gives $b = 1 + 1/2 = 3/2$; hence the line is

$$y = -\frac{x}{2} + \frac{3}{2} \quad (5.19)$$

Definition 5.2 The derivative of a function f at a number a is defined as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (5.20)$$

We will sometimes use the Δ operator to indicate a small **increment**; for example as x goes from x_1 to x_2 , the change in x is denoted as

$$\Delta x = x_2 - x_1 \quad (5.21)$$

If $y = f(x)$, the corresponding change in y is denoted by

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) \quad (5.22)$$

Then the **average rate of change** of y with respect to x over the interval $[x_1, x_2]$ is given by the slope of the secant line:

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (5.23)$$

The **instantaneous rate of change** at $x = x_1$ is the limit as $x_2 \rightarrow x_1$ of $\Delta y/\Delta x$:

$$\text{instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1) \quad (5.24)$$

In other words, the instantaneous rate of change of a function at the number a is the same thing as the derivative of the function at $x = a$.

Example 5.3 (Text 3.1.14) If a rock is thrown upwards on Marx with a velocity of 10 meters/second, its height in meters after t seconds is given by

$$H(t) = 10t - 1.86t^2 \quad (5.25)$$

(a) Find the velocity of the rock after 1 second.

At $t = 1$, the height is $H(1) = 10(1) - 1.86(1^2) = 8.14$ meters. The velocity is the

instantaneous rate of change, or $H'(1)$. So

$$H'(1) = \lim_{h \rightarrow 0} \frac{H(1+h) - H(1)}{h} \quad (5.26)$$

$$= \lim_{h \rightarrow 0} \frac{10(1+h) - 1.86(1+h)^2 - 8.14}{h} \quad (5.27)$$

$$= \lim_{h \rightarrow 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 8.14}{h} \quad (5.28)$$

$$= \lim_{h \rightarrow 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 8.14}{h} \quad (5.29)$$

$$= \lim_{h \rightarrow 0} \frac{10h - 3.72h - 1.86h^2}{h} \quad (5.30)$$

$$= \lim_{h \rightarrow 0} (10 - 3.72 - 1.86h) = 6.28 \quad (5.31)$$

(b) Find the velocity of the rock when $t = a$.

$$H'(a) = \lim_{h \rightarrow 0} \frac{H(a+h) - H(a)}{h} \quad (5.32)$$

$$= \lim_{h \rightarrow 0} \frac{10(a+h) - 1.86(a+h)^2 - 10a - 1.86a^2}{h} \quad (5.33)$$

$$= \lim_{h \rightarrow 0} \frac{10a + 10h - 1.86(a + 2ah + h^2) - 10a - 1.86a^2}{h} \quad (5.34)$$

$$= \lim_{h \rightarrow 0} \frac{10a + 10h - 1.86a - 3.72ah - 1.86h^2 - 8.14a}{h} \quad (5.35)$$

$$= \lim_{h \rightarrow 0} \frac{10h - 3.72ah - 1.86h^2}{h} \quad (5.36)$$

$$= \lim_{h \rightarrow 0} (10 - 3.72a - 1.86h) = 10 - 3.72a \quad (5.37)$$

(c) When will the rock hit the surface?

The rock will hit the surface when $H = 0$. This means

$$0 = 10t - 1.86t^2 = t(10 - 1.86t) \implies t = 10/1.86 = 5.37634 \quad (5.38)$$

(d) How fast will the rock be moving when it hits the surface?

$$H'(5.37634) = 10 - 3.72(5.37634) = -9.99998 \text{ meters/second} \quad (5.39)$$

Topic 6

The Derivative as a Function

We now can extend our definition of a derivative at a point to a function.

Definition 6.1 Let $f(x)$ be a function. Then the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (6.1)$$

is called **the derivative of the function** $f(x)$, if the limit exists, and we say the function $f(x)$ is **differentiable** at x . If the limit does not exist, we say the function is **not differentiable** at x .

Calculating the derivative of a function is similar to calculating the derivative at a point.

Example 6.1 Let $f(x) = x$. Find $f'(x)$ Since $f(x) = x$, $f(x+h) = x+h$. Hence

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h-x}{h} = \frac{h}{h} = 1 \quad (6.2)$$

$$\implies \lim_{x \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \rightarrow 0} (1) = 1 \quad (6.3)$$

Example 6.2 Let $f(x) = x^2$. Find $f'(x)$.

Since $f(x) = x^2$, then $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$; hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \quad (6.4)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 - x^2)}{h} \quad (6.5)$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \quad (6.6)$$

$$= \lim_{h \rightarrow 0} (2x + h) = 2x \quad (6.7)$$

Thus the derivative of the function $f(x) = x^2$ is the function $f'(x) = 2x$. This tells us that the slope of $y = x^2$ has a value of $2x$ at any number x .

Example 6.3 Let $f(x) = \sqrt{x}$. Find $f'(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (6.8)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad (6.9)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \quad (6.10)$$

$$= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \quad (6.11)$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \quad (6.12)$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad (6.13)$$

$$= \frac{1}{2\sqrt{x}} \quad (6.14)$$

Notation. Let $y = f(x)$. Then the following notations are equivalent ways of

representing $f'(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x) \quad (6.15)$$

Definition 6.2 A function f is **differentiable** at a if $f'(a)$ exists. It is **differentiable on the open interval** (a, b) if $f'(x)$ exists for every $x \in (a, b)$.

Theorem 6.3 If $f(x)$ is differentiable at a then $f(x)$ is continuous at a .

Proof. Assume that $f(x)$ is differentiable at a . We need to show that

$$\lim_{x \rightarrow a} f(x) = f(a) \quad (6.16)$$

because this proves that that f is continuous at a .

But since f is differentiable at a , we know that the derivative exists and the following limit exists:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (6.17)$$

By the laws of limits,

$$\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} (f(x) - f(a)) \times \frac{x - a}{x - a} \quad (6.18)$$

$$= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} (x - a) \quad (6.19)$$

$$= \left(\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right) \times \left(\lim_{x \rightarrow a} (x - a) \right) \quad (6.20)$$

$$= f'(a) \lim_{x \rightarrow a} (x - a) \quad (6.21)$$

$$= (f'(a)) \times (0) = 0 \quad (6.22)$$

Hence

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [f(x) - f(a) + f(a)] \quad (6.23)$$

$$= \left[\lim_{x \rightarrow a} (f(x) - f(a)) \right] + \left[\lim_{x \rightarrow a} f(a) \right] \quad (6.24)$$

$$= 0 + f(a) = f(a) \quad (6.25)$$

which is exactly equation 6.16, which is what we wanted to prove. Hence $f(x)$ is continuous at a . \square

Definition 6.4 Higher Order Derivatives. Let $y = f(x)$. We define the **second derivative** y'' as

$$f''(x) = \frac{d}{dx} f'(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \quad (6.26)$$

if the limit exists. Similarly, the third derivative is given by

$$f'''(x) = \frac{d}{dx} f''(x) \quad (6.27)$$

and so forth.

Example 6.4 Let $y = x^2$. Find y' , y'' , and y''' .

In example 6.2 we found that $y' = 2x$.

$$y'' = \lim_{h \rightarrow 0} \frac{y'(x+h) - y'(x)}{h} \quad (6.28)$$

$$= \lim_{h \rightarrow 0} \frac{2(x+h) - 2x}{h} \quad (6.29)$$

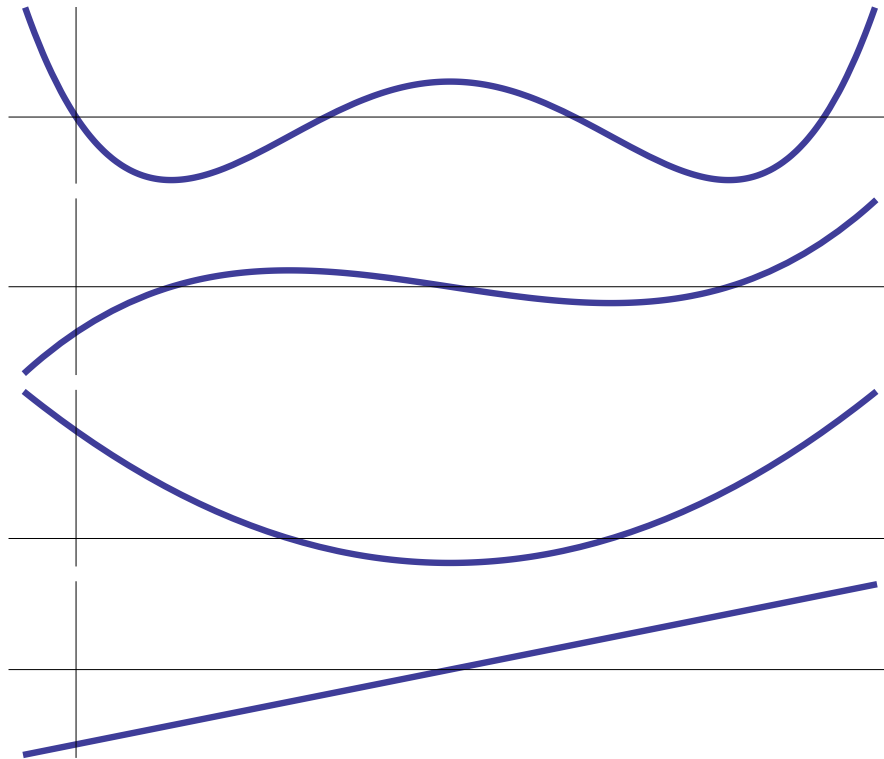
$$= \lim_{h \rightarrow 0} \frac{2h}{h} = 2 \quad (6.30)$$

and similarly

$$y'' = \lim_{h \rightarrow 0} \frac{y''(x+h) - y''(x)}{h} \quad (6.31)$$

$$= \lim_{h \rightarrow 0} \frac{2 - 2}{h} = 0 \quad (6.32)$$

The following figure illustrates a function and its first three derivatives. It is easy to sketch each derivative because the points where the function has a horizontal tangent line have a zero derivative.



Topic 7

Formulas for Derivatives

In this section we will derive formulas for derivatives of several classes of functions.

Theorem 7.1 Derivative of a Constant. Let c be a constant. Then

$$\frac{d}{dx}(c) = 0 \quad (7.1)$$

Proof. Let $f(x) = c$; then $f(x + h) = c$. Hence

$$\frac{d}{dx}(c) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0 \quad \square \quad (7.2)$$

Theorem 7.2 Derivative of Power Functions

$$\frac{d}{dx}x = 1, \quad \frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}x^3 = 3x^2, \dots \quad \frac{d}{dx}x^n = nx^{n-1} (n \neq -1) \quad (7.3)$$

Proof. The first two formulas were derived in examples 6.1 and 6.2. Since the third formula is a special case of the n^{th} formula, we will prove the formula for n . We start

by proving the following algebraic relation:

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) \quad (7.4)$$

The right hand side can be expanded by multiplying out:

$$(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) \quad (7.5)$$

$$= x(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) - \\ a(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) \quad (7.6)$$

$$= x^n + x^{n-1}a + x^{n-2}a^2 + \cdots + x^3a^{n-3} + x^2a^{n-2} + xa^{n-1} \\ - x^{n-1}a - x^{n-2}a^2 - x^{n-3}a^3 - \cdots - x^2a^{n-2} - xa^{n-1} - a^n \quad (7.7)$$

$$= x^n - a^n \quad (7.8)$$

which verifies equation 7.4.

Now let $f(x) = x^n$. Then by equation 7.4,

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1} \quad (7.9)$$

and therefore

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (7.10)$$

$$= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) \quad (7.11)$$

$$= a^{n-1} + a^{n-2} \times a + \cdots + a^2 \times a^{n-3} + a \times a^{n-2} + a^{n-1} \quad (7.12)$$

$$= na^{n-1} \quad (7.13)$$

and therefore $f'(x) = nx^{n-1}$. □

Theorem 7.3 Constant Multiple of a Function. Let $f(x)$ be a differentiable function and c a constant. Then

$$\frac{d}{dx} [cf(x)] = c \frac{d}{dx} f(x) \quad (7.14)$$

Proof. Let $g(x) = cf(x)$; then $g(x+h) = cf(x+h)$ and consequently,

$$\frac{d}{dx} [cf(x)] = g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad (7.15)$$

$$= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \quad (7.16)$$

$$= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = cf'(x) \quad \square \quad (7.17)$$

Theorem 7.4 Sum and Difference Rules. If $f(x)$ and $g(x)$ are differentiable then

$$\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x) \quad (7.18)$$

Proof. Let $p(x) = f(x) + g(x)$. Then $p(x+h) = f(x+h) + g(x+h)$; hence

$$\frac{d}{dx} [f(x) + g(x)] = p'(x) = \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} \quad (7.19)$$

$$= \lim_{h \rightarrow 0} \left[\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \right] \quad (7.20)$$

$$= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \quad (7.21)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad (7.22)$$

$$= f'(x) + g'(x) \quad \square \quad (7.23)$$

Theorem 7.5 Product Rule. If $f(x)$ and $g(x)$ are differentiable then

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \quad (7.24)$$

Proof. Let $p(x) = f(x)g(x)$. Then $p(x+h) = f(x+h)g(x+h)$.

$$\frac{p(x+h) - p(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \quad (7.25)$$

$$= \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \quad (7.26)$$

$$= f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \quad (7.27)$$

Hence

$$(fg)' = p' = \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} \quad (7.28)$$

$$= \lim_{h \rightarrow 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x)\frac{f(x+h) - f(x)}{h} \quad (7.29)$$

$$= \lim_{h \rightarrow 0} f(x+h) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (7.30)$$

$$= f(x)g'(x) + g(x)f'(x) \quad \square \quad (7.31)$$

Example 7.1 Let $p(x) = (x^2 + 3x)(4x^{17} + 6x^{43})$. Find $p'(x)$.

We can write $p = fg$, where

$$f(x) = x^2 + 3x, \quad g(x) = 4x^{17} + 6x^{43} \quad (7.32)$$

Differentiating,

$$f' = 2x + 3, \quad g' = 4(17)x^{16} + 6(43)x^{42} = 68x^{16} + 258x^{42} \quad (7.33)$$

and therefore

$$p'(x) = fg' + gf' = (x^2 + 3x)(68x^{16} + 258x^{42}) + (4x^{17} + 6x^{43})(2x + 3) \quad (7.34)$$

Theorem 7.6 Quotient Rule. If $f(x)$ and $g(x)$ are differentiable, then at points where $g(x) \neq 0$,

$$\left[\frac{f}{g}\right]' = \frac{gf' - fg'}{g^2} \quad (7.35)$$

Proof. Let $q(x) = f(x)/g(x)$. Then

$$q(x+h) = \frac{f(x+h)}{g(x+h)} \quad (7.36)$$

hence

$$\frac{q(x+h) - q(x)}{h} = \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \quad (7.37)$$

$$= \frac{g(x)f(x+h) - f(x)g(x+h)}{hg(x)g(x+h)} \quad (7.38)$$

$$= \frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - f(x)g(x+h)}{hg(x)g(x+h)} \quad (7.39)$$

$$= \frac{g(x)f(x+h) - g(x)f(x)}{hg(x)g(x+h)} + \frac{g(x)f(x) - f(x)g(x+h)}{hg(x)g(x+h)} \quad (7.40)$$

$$= g(x) \frac{f(x+h) - f(x)}{hg(x)g(x+h)} + f(x) \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \quad (7.41)$$

$$= \frac{1}{g(x+h)} \frac{f(x+h) - f(x)}{h} - \frac{f(x)}{g(x)g(x+h)} \frac{g(x+h) - g(x)}{h} \quad (7.42)$$

$$\rightarrow \frac{f'}{g} - \frac{f}{g^2}g' = \frac{gf' - fg'}{g^2} \text{ as } h \rightarrow 0 \quad (7.43)$$

Example 7.2 Find $q'(x)$ for

$$q(x) = \frac{4x^3 - 3x}{2x + 17} \quad (7.44)$$

Since $q = f/g$ where $f = 4x^3 - 3x$ and $g = 2x + 17$, we have

$$f' = 12x - 3, \quad g' = 2 \quad (7.45)$$

and therefore

$$q' = \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} = \frac{(2x + 17)(12 + 3) - (4x^3 - 3x)(2)}{(2x + 17)^2} \quad (7.46)$$

Remark 7.7 Let $n > 0$ be a positive integer. Then

$$\frac{d}{dx}x^{-n} = -nx^{-n-1} \quad (7.47)$$

In other words, the power law also holds for negative integer powers. *Proof.* Use the quotient rule:

$$x^{-n} = \frac{1}{x^n} = \frac{f}{g} \quad (7.48)$$

where $f = 1$ and $g = x^n$. But $f' = 0$ and $g' = nx^{n-1}$; hence

$$\frac{d}{dx}x^{-n} = \frac{gf' - fg'}{g^2} = \frac{x^n \times (0) - (1) \times nx^{n-1}}{(x^n)^2} = -nx^{-n-1} \quad \square \quad (7.49)$$

Remark 7.8 Generalized Power Rule. If n is any real number (not just an integer), then

$$(x^n)' = nx^{n-1} \quad (7.50)$$

The proof is a little bit beyond our abilities right now.

Differentiation Formulas

Rule	$f(x)$	$f'(x)$
Constant Rule	c	0
Power Rule, $n = 1$	x	1
Power Rule	x^n	nx^{n-1}
Constant Multiple	$cf(x)$	$cf'(x)$
Sum/Difference Rule	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
Product Rule	fg	$fg' + gf'$
Quotient Rule	f/g	$\frac{gf' - fg'}{g^2}$

Topic 8

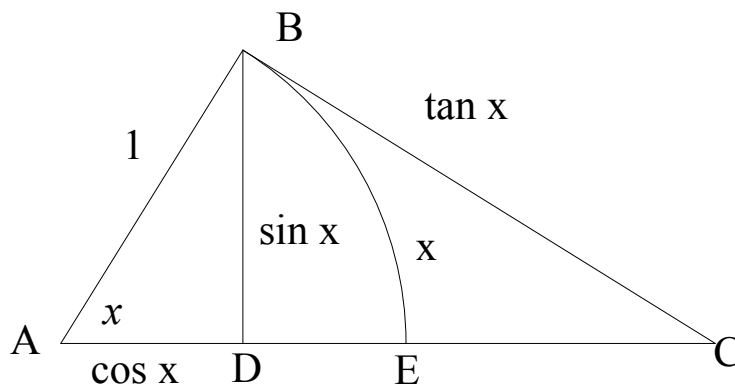
Derivatives of Trigonometric Functions

We now prove the following identity that we have already studied:

Theorem 8.1

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (8.1)$$

Proof. The geometry is illustrated in the following figure:



Point A is the center of a unit circle; arc BE is an arc of that circle; and BD is a

perpendicular segment dropped from B to line AC . Then

$$\sin x = \frac{BD}{AB} = BD \quad (8.2)$$

because $AB = 1$ is a radius. Thus the arc-length BE is

$$\text{arclength } BE = \text{radius} \times \text{angle } x = x \quad (8.3)$$

in radians because the radius is 1.

From the figure we see that

$$\text{length of segment } BD < \text{length of arc } BE \implies \sin x < x \quad (8.4)$$

The line BC is constructed so it is tangent to the circle at B ; hence triangle ABC is a right triangle with right angle at B . This means that

$$\tan x = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{BC}{AB} = BC \quad (8.5)$$

where the last step follows because $AB = 1$.

Also from the picture, we see that

$$x < \tan x \quad (8.6)$$

Combining this with equation 8.4 gives

$$\sin x < x < \tan x = \frac{\sin x}{\cos x} \quad (8.7)$$

Dividing across by $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad (8.8)$$

Using the identity

$$a < b \implies \frac{1}{b} < \frac{1}{a} \quad (8.9)$$

we get

$$\cos x < \frac{\sin x}{x} < 1 \quad (8.10)$$

But

$$\lim_{x \rightarrow 0} \cos x = 1, \quad \lim_{x \rightarrow 0} 1 = 1 \quad (8.11)$$

Hence by the pinching theorem

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (8.12)$$

which proves our identity. \square

Example 8.1

$$\lim_{x \rightarrow 0} x \csc x = \lim_{x \rightarrow 0} x \frac{\cos x}{\sin x} \quad (8.13)$$

$$= \left(\lim_{x \rightarrow 0} \frac{x}{\sin x} \right) \left(\lim_{x \rightarrow 0} \cos x \right) \quad (8.14)$$

$$= \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = 1 \quad \square \quad (8.15)$$

Example 8.2

$$\lim_{x \rightarrow 0} \frac{\sin 7x}{4x} = \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \times \frac{7}{4} = \frac{7}{4} \quad \square \quad (8.16)$$

Theorem 8.2

$$\boxed{\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0} \quad (8.17)$$

Proof.

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \times \frac{\cos x + 1}{\cos x + 1} \quad (8.18)$$

$$= \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{x(1 + \cos x)} \quad (8.19)$$

$$= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x(1 + \cos x)} \quad (8.20)$$

$$= -\lim_{x \rightarrow 0} \frac{\sin x}{x} \times \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + 1} \quad (8.21)$$

$$= (-1) \times \frac{0}{1 + 0} = 0 \quad \square \quad (8.22)$$

□

Theorem 8.3

$$\boxed{\frac{d}{dx} \sin x = \cos x} \quad (8.23)$$

Proof. Let $f(x) = \sin x$. Then

$$f(x + h) = \sin(x + h) \quad (8.24)$$

$$= \sin x \cos h + \cos x \sin h \quad (8.25)$$

$$\frac{f(x + h) - f(x)}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \quad (8.26)$$

$$= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \quad (8.27)$$

$$= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \quad (8.28)$$

Therefore

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (8.29)$$

$$= \lim_{h \rightarrow 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] \quad (8.30)$$

$$= \lim_{h \rightarrow 0} \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \quad (8.31)$$

$$= (1) \times \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + (\cos x) \times (1) \quad (8.32)$$

$$= \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \quad (8.33)$$

$$= \cos x \quad (8.34)$$

hence $f'(x) = \cos x$ as desired. □

By a similar argument we have

Theorem 8.4

$$\boxed{\frac{d}{dx} \cos x = -\sin x}$$

We can find the derivative of the tan using the quotient rule, since

$$\tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)} \quad (8.35)$$

where $f(x) = \sin x$ and $g(x) = \cos x$. Thus

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{gf' - fg'}{g^2} = \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x} \quad (8.36)$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad (8.37)$$

Theorem 8.5

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

Summary of Trigonometric Derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Example 8.3 Find the derivative of $y = \frac{1 + \sin x}{x + \cos x}$

This is a quotient f/g with $f(x) = 1 + \sin x$ and $g(x) = x + \cos x$, Hence

$$f'(x) = \cos x \text{ and } g'(x) = 1 - \sin x \quad (8.38)$$

Thus

$$\frac{d}{dx} \frac{1 + \sin x}{x + \cos x} = \frac{gf' - fg'}{g^2} \quad (8.39)$$

$$= \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} \quad (8.40)$$

$$= \frac{x \cos x + \cos^2 x - 1 + \sin^2 x}{(x + \cos x)^2} \quad (8.41)$$

$$= \frac{x \cos x}{(x + \cos x)^2} \quad \square \quad (8.42)$$

Topic 9

The Chain Rule

Theorem 9.1 Let $g(x)$ be differentiable at x and $f(u)$ be differentiable at $u = g(x)$.

Then

$$\boxed{\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)} \quad (9.1)$$

Another way of writing this is

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}} \quad (9.2)$$

Example 9.1 Find $f'(x)$ for $f(x) = \sqrt{3x^2 + 2}$ We can begin by making a substitution

$$u(x) = 3x^2 + 2 \quad (9.3)$$

Then

$$f(x) = \sqrt{3x^2 + 2} = \sqrt{u(x)} \quad (9.4)$$

Hence

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \frac{du}{dx} \quad (9.5)$$

Since

$$\frac{df}{du} = \frac{d}{du} \sqrt{u} = \frac{d}{du} u^{1/2} = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}} \quad (9.6)$$

and

$$\frac{du}{dx} = \frac{d}{dx} (3x^2 + 2) = 6x \quad (9.7)$$

we end up with

$$f'(x) = \frac{1}{2\sqrt{u}} \times (6x) = \frac{3x}{\sqrt{3x^2 + 2}} \quad (9.8)$$

Example 9.2 Let $y = \cos(x^2)$. Find dy/dx .

We begin by substituting $u = x^2$. Then $y = \cos u$ and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad (9.9)$$

$$= \left(\frac{d}{du} \cos u \right) \times \left(\frac{d}{dx} x^2 \right) \quad (9.10)$$

$$= (-\sin u) \times (2x) \quad (9.11)$$

$$= -2x \sin(x^2) \quad (9.12)$$

Example 9.3 Let $y = (\cos x)^2$. Find dy/dx .

Let $u = \cos x$; then $y = u^2$, and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad (9.13)$$

$$= \left(\frac{d}{du} u^2 \right) \times \left(\frac{d}{dx} \cos x \right) \quad (9.14)$$

$$= -2u \sin x \quad (9.15)$$

$$= -2 \cos x \sin x \quad (9.16)$$

If we apply the chain rule to find $(d/dx)(f(x)^n)$ we obtain the following.

Theorem 9.2 Chain Rule for Powers

$$\boxed{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}} \quad (9.17)$$

or equivalently,

$$\boxed{\frac{d}{dx}(f(x))^n = nf(x)^{n-1}f'(x)} \quad (9.18)$$

Example 9.4 Differentiate $y = (4x^7 - 3x^6)^{427}$.

By the chain rule for powers,

$$\frac{d}{dx}((4x^7 - 3x^6)^{427}) = 427(4x^7 - 3x^6)\frac{d}{dx}(4x^7 - 3x^6) \quad (9.19)$$

$$= 427(4x^7 - 3x^6)(28x^6 - 18x^5) \quad (9.20)$$

Example 9.5 Differentiate $f(x) = \frac{1}{\sqrt{x^2 + 99}}$

$$\frac{d}{dx}\frac{1}{\sqrt{x^2 + 99}} = \frac{d}{dx}(x^2 + 99)^{-1/2} \quad (9.21)$$

$$= -\frac{1}{2}(x^2 + 99)^{-3/2}\frac{d}{dx}(x^2 + 99) \quad (9.22)$$

$$= -\frac{x}{(x^2 + 99)^{3/2}} \quad (9.23)$$

Example 9.6 Let $y = \sin(x \cos x)$. Find y' . If we define $u = x \cos x$, then $y = \sin u$.

Hence

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad (9.24)$$

$$= \left(\frac{d}{du}\sin u\right) \times \left(\frac{d}{dx}[x \cos x]\right) \quad (9.25)$$

By the product rule,

$$\frac{d}{dx}x \cos x = (x)(\cos x)' + (\cos x)(x)' = -x \sin x + \cos x \quad (9.26)$$

and thus

$$y' = \cos u(-x \sin x + \cos x) = [\cos(x \cos x)](-x \sin x + \cos x) \quad (9.27)$$

Example 9.7 Find dy/dx for

$$y = \frac{x}{\sqrt{7-3x}} \quad (9.28)$$

For this problem we have to first apply the chain rule, then apply the quotient rule.

We have a quotient f/g where $f = x$ and $g = \sqrt{7-3x}$. Hence $f' = 1$ and by the chain rule:

$$\frac{d}{dx}\sqrt{7-3x} = \frac{1}{2}(7-3x)^{-3/2} \frac{d}{dx}(7-3x) = \frac{-3}{2(7-3x)^{3/2}} \quad (9.29)$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{d}{dx} \frac{f}{g} = \frac{gf' - fg'}{g^2} \quad (9.30)$$

$$= \frac{(\sqrt{7-3x})(1) - (x)(-3/[2(7-3x)^{3/2}])}{7-3x} \quad (9.31)$$

$$= \frac{(7-3x)^2 + 3x/2}{(7-3x)^{5/2}} \quad (9.32)$$

Topic 10

Implicit Differentiation

Sometimes we cannot get an explicit formula for the function we want to differentiate.

For example, suppose that

$$x^3 + y^3 = 10xy \tag{10.1}$$

How do find y' from this?

The basic method of implicit differentiation is based on the observation that if

$$f(x) = g(x) \tag{10.2}$$

then

$$f'(x) = g'(x) \tag{10.3}$$

In other words, we differentiate both sides of the equation and then solve for y' .

Example 10.1 Find y' if $x^3 + y^3 = 10xy$.

$$\frac{d}{dx} (x^3 + y^3) = \frac{d}{dx} (10xy) \tag{10.4}$$

The main trick in implicit differentiation is to remember that whenever we see something like

$$\frac{dy}{dx} \tag{10.5}$$

we replace it by y' and then leave it that way in the equation. Hopefully at the end of the process we will end up with an equation in which we can solve for y' as a function of x and y .

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 10 \left[x \frac{dy}{dx} + y \frac{dx}{dx} \right] \tag{10.6}$$

$$3x^2 + 3y^2y' = 10xy' + 10y \tag{10.7}$$

$$3y^2y' - 10xy' = 10y - 3x^2 \tag{10.8}$$

$$y'(3y^2 - 10x) = 10y - 3x^2 \tag{10.9}$$

$$y' = \frac{10y - 3x^2}{3y^2 - 10x} \tag{10.10}$$

Example 10.2 Find y' when $2\sqrt{x} + \sqrt{y} = 3$.

$$\frac{d}{dx}2\sqrt{x} + \frac{d}{dx}\sqrt{y} = \frac{d}{dx}(3) \tag{10.11}$$

$$\frac{d}{dx}(2x^{1/2}) + \frac{d}{dx}(y^{1/2}) = 0 \tag{10.12}$$

$$2(1/2)(x^{-1/2}) + (1/2)(y^{-1/2})y' = 0 \tag{10.13}$$

$$y = -2\sqrt{\frac{y}{x}} \tag{10.14}$$

Example 10.3 Find the equation of the tangent line to the hyperbola

$$x^2 + 2xy - y^2 + x = 2 \tag{10.15}$$

at the point (1,2). Using implicit differentiation,

$$\frac{d}{dx}x^2 + 2\frac{d}{dx}(xy) - \frac{d}{dx}y^2 + \frac{d}{dx}(x) = \frac{d}{dx}(2) \quad (10.16)$$

$$2x + 2[xy' + y] - 2yy' + 1 = 0 \quad (10.17)$$

When $x = 1$ and $y = 2$ this means

$$2(1) + 2[(1)y' + 2] - 2(2)y' + 1 = 0 \quad (10.18)$$

$$2 + 2y' + 4 - 4y' + 1 = 0 \quad (10.19)$$

$$7 = 2y' \quad (10.20)$$

Hence the slope at (1,2) is

$$m = y' = \frac{7}{2} \quad (10.21)$$

The equation of the line is $y = mx + b$ hence

$$2 = (7/2)(1) + b \implies b = 2 - (7/2) = -3/2 \implies y = \frac{7}{2}x - \frac{3}{2} \quad (10.22)$$

Example 10.4 Find all the points on the curve

$$x^2y^2 + xy = -2 \quad (10.23)$$

where the slope of the tangent line is -1.

Differentiating implicitly,

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(-2) \quad (10.24)$$

Using the product rule in each of the terms on the left,

$$x^2 \frac{d}{dx} y^2 + y^2 \frac{d}{dx} x^2 + x \frac{dy}{dx} + y \frac{dx}{dx} = 0 \quad (10.25)$$

Since $dy/dx = y'$ and $(d/dx)y^2 = 2yy'$

$$x^2(2yy') + y^2(2x) + xy' + y = 0 \quad (10.26)$$

$$y'(2x^2y + x) + 2xy^2 + y = 0 \quad (10.27)$$

We are not actually interested in y' but rather in the places where $y' = -1$. Substituting $y' = -1$ gives

$$-2x^2y - x + 2xy^2 + y = 0 \quad (10.28)$$

$$(2x)y^2 + (1 - 2x^2)y - x = 0 \quad (10.29)$$

By the quadratic equation, we can solve for y :

$$y = \frac{2x^2 - 1 \pm \sqrt{(1 - 2x^2)^2 - 4(2x)(-x)}}{4x} \quad (10.30)$$

$$y = \frac{(2x^2 - 1) \pm \sqrt{(2x^2 + 1)^2}}{4x} \quad (10.31)$$

$$y = \frac{(2x^2 - 1) + (2x^2 + 1)}{4x} \text{ or } \frac{(2x^2 - 1) - (2x^2 + 1)}{4x} \quad (10.32)$$

$$y = x \text{ or } y = -\frac{1}{2x} \quad (10.33)$$

Topic 11

Related Rates

In a “related rates” problem, two (or more) variables are changing with respect to time, and both variables are related to one-another.

The general idea is to express the problem in terms of whichever variable is easier to use, and then use the chain rule.

Example 11.1 Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{sec}$. How fast is the radius changing when the diameter is 50 cm?

Defining our variables: $V =$ volume and $r =$ radius.

We want to find dr/dt when $r = 50/2 = 25$.

We are given that

$$\frac{dV}{dt} = 100 \tag{11.1}$$

and that the balloon is spherical, hence

$$V = \frac{4}{3}\pi r^3 \tag{11.2}$$

By the chain rule,

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) = 4\pi r^2 \frac{dr}{dt} \quad (11.3)$$

Thus at all time (using $dV/dt = 100$),

$$100 = 4\pi r^2 \frac{dr}{dt} \quad (11.4)$$

and when $r = 25$ we get

$$100 = 4(25)^2 \pi \frac{dr}{dt} \quad (11.5)$$

Hence when $r = 25$,

$$\frac{dr}{dt} = \frac{100}{4(25)(25)\pi} = \frac{1}{25\pi} \quad (11.6)$$

Topic 12

Infinitesimals

The **Principle of Archimedes** says that for any two nonzero numbers x and M , such that $x < M$, if you add x to itself enough times you will get a number larger than M , i.e., for some positive n ,

$$\underbrace{x + x + \cdots + x}_{n \text{ times}} > M \quad (12.1)$$

All real numbers and all of traditional mathematics obeys the principle of Archimedes. An equivalent way of stating the principle of Archimedes is that there is some real number n such that

$$nx > M \quad (12.2)$$

This principle is so ingrained into our mathematical subconscious (if such a thing even exists) that we rarely even bother to think about it.

Leibniz was a contemporary of Newton who also invented Calculus. Unfortunately, Newton got the credit, because Leibniz used a different approach. Instead of using limits, Leibniz's idea was to define a new number that is not an element of the real

numbers, but which can be added to real numbers to create a new set of numbers, as well as a new type of mathematics. This new number was called an **infinitesimal**. An infinitesimal is a number ∂ so small that no matter how many times you add it to something else you never reach even the smallest real number. In other words, ∂ violates the principle of Archimedes, and consequently, ∂ is a positive number that is smaller than the smallest real number.

Some calculus textbooks say that the principle of an infinitesimal is vague. This is not at all true. Infinitesimals can be defined with just as much mathematical rigour as limits; unfortunately, they are rarely mentioned in introductory calculus and even advanced calculus and analysis textbooks.

You are allowed to do all of the normal arithmetical operations on infinitesimals, as well as between infinitesimals and real numbers, and the answers come out just as you might expect:

$$\partial + \partial = 2\partial \tag{12.3}$$

$$\partial + \partial + \partial = 3\partial \tag{12.4}$$

$$\partial \times \partial = (\partial)^2 \tag{12.5}$$

and so forth. The last result, ∂^2 , is called, a **second-order infinitesimal** because it is smaller than any first-order infinitesimal. You can keep adding second-order infinitesimals together as long as you like and you will never reach a first-order infinitesimal. Thus

$$3\partial + \partial^2 > 3\partial \tag{12.6}$$

but there is no real number n such that

$$3\partial + n\partial^2 > 4\partial \tag{12.7}$$

And so on. Similarly, you can add an infinitesimal to a real number. The result is called a **hyperreal** number, such as

$$x = 17 + 3\partial \tag{12.8}$$

Since the sum of a real number and an infinitesimal is a hyperreal very close to some real number, we can define an operator

$$R(x) = \text{nearest real number to } x \tag{12.9}$$

that has the following two properties:

$$R(x + y) = R(x) + R(y) \tag{12.10}$$

$$R(xy) = R(x)R(y) \tag{12.11}$$

By definition of ∂ and R ,

$$R(\partial) = 0 \tag{12.12}$$

Hence

$$R(12 - 3\partial + 18\partial^2) = R(12) - R(3\partial) + 18(\partial^2) \tag{12.13}$$

$$= 12 - 3R(\partial) + 18R(\partial)R(\partial) \tag{12.14}$$

$$= 12 - 3(0) + 18(0)(0) \tag{12.15}$$

$$= 12 \tag{12.16}$$

Leibniz defined the tangent line of a curve as the line that goes through two points that are infinitesimally separated from one another along the hyperreal x and y axes,

with hyperreal slope

$$m = \frac{f(x + \partial) - f(x)}{\partial} \quad (12.17)$$

Since the number m is not a real but is a hyperreal, the **Leibniz Derivative** is defined as

$$\boxed{f'(x) = R\left(\frac{f(x + \partial) - f(x)}{\partial}\right)} \quad (12.18)$$

Example 12.1 Calculate $f'(x)$ for $f(x) = 5x^2 + 3x + 15$. We have

$$f(x + \partial) = 5(x + \partial)^2 + 3(x + \partial) + 15 \quad (12.19)$$

$$= 5x^2 + 10x\partial + 5\partial^2 + 3x + 3\partial + 15 \quad (12.20)$$

Hence

$$\begin{aligned} f(x + \partial) - f(x) &= 5x^2 + 10x\partial + 5\partial^2 + 3x + 3\partial + 15 \\ &\quad - 5x^2 - 3x - 15 \end{aligned} \quad (12.21)$$

$$= 10x\partial + 5\partial^2 + 3\partial \quad (12.22)$$

$$= \partial(10x + 5\partial + 3) \quad (12.23)$$

and consequently the derivative is

$$f'(x) = R\left(\frac{f(x + \partial) - f(x)}{\partial}\right) \quad (12.24)$$

$$= R(10x + 5\partial + 3) \quad (12.25)$$

$$= 10x + 3 \quad (12.26)$$

Thus we can derive a formula for the derivative without using limits at all. In fact,

virtually every known fact about calculus (this includes integral calculus and partial derivatives that we will do next semester and next year in Math 150B and Math 250) can be derived using the Leibniz definition rather than the Newton, and without EVER calculating any limits!

Leibniz did not use the symbol ∂ . Instead, he defined an infinitesimal in x using an operator called the delta-operator, written as Δx , read as “delta-x.” This is somewhat confusing because it looks like two numbers Δ and x multiplied together but it is not. The number Δx is a single number, that is in fact an infinitesimal. This is the origin of the notation Δx for a very small number, used by our textbook and most other calculus books. Leibniz’s heritage is that we use the notation dy/dx to represent a derivative, even though most mathematicians do not even understand where the origin of the notation, and just think of Δx as a “small” number, but not as an infinitesimal.

Topic 13

Differentials

We can define a **differential** in x as a very small number (or in the Leibniz notation, an infinitesimal) and a **differential in y** as the difference

$$dy = f(x + dx) - f(x) \tag{13.1}$$

where dx is a differential in x . Then in the Leibniz notation,

$$f'(x) = R\left(\frac{dy}{dx}\right) = R\left(\frac{f(x + dx) - f(x)}{dx}\right) \tag{13.2}$$

and the in the Newton notation

$$f'(x) = \lim_{dx \rightarrow 0} \frac{dy}{dx} = \lim_{dx \rightarrow 0} \frac{f(x + dx) - f(x)}{dx} \tag{13.3}$$

In the first case, dx is an infinitesimal; in the second case it is a very small real number. Formally these are quite different; but in practice we tend to ignore the difference,

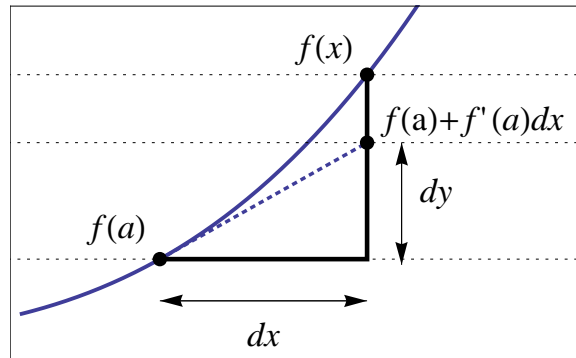
because both formalisms give rise to the notation

$$f'(x) = \frac{dy}{dx} \quad (13.4)$$

where dx and dy signify some kind of changes in x and y respectively. In fact, using equation 13.4 to designate a derivative, either notation leads us to conclude that

$$\boxed{dy = \frac{dy}{dx} dx} \quad (13.5)$$

which is most commonly taken as the definition of a differential.



We can use the differential to create a **linear approximation** to a function at x for x near a in the sense that

$$x = a + dx \quad (13.6)$$

by writing

$$f(x) = f(a) + df = f(a) + f'(a)dx = f(a) + f'(a)(x - a) \quad (13.7)$$

We could also have obtained the same result by approximating

$$f'(x) \approx \frac{f(x) - f(a)}{x - a} \quad (13.8)$$

and then solving for $f(x)$.

Example 13.1 Find dy when $y = (2x + 3)^{-4}$ From the chain rule we have

$$f'(x) = (-4)(2x + 3)^{-5}(2) = \frac{-8}{(2x + 3)^5} \quad (13.9)$$

and therefore

$$dy = f'(x)dx = \frac{-8dx}{(2x + 3)^5} \quad \square \quad (13.10)$$

Example 13.2 Estimate the change dy in the function $y = x^2$ as x increases from 10 to 10.1 and the compare it with the exact change Δy . We can estimate the change as

$$dy = f'(x)dx = 2xdx \quad (13.11)$$

At $x = 10$ and $dx = 10.1 - 10 = 0.1$ we have

$$dy = 2(10)(0.1) = 2.0 \quad (13.12)$$

The actual change is

$$\Delta y = f(10.1) - f(10) = 10.1^2 - 10^2 = 102.01 - 100 = 2.01 \quad \square \quad (13.13)$$

Example 13.3 Estimate $\sqrt{4.2}$

Let $f(x) = x^{1/2}$. Then $f'(x) = (1/2)x^{-1/2}$. Since $4.2 = 4 + 0.2$, we can let

$$a = 4, \quad dx = 0.2 \implies f(a) = \sqrt{4} = 2, \quad f'(a) = \frac{1}{2\sqrt{4}} = 0.25 \quad (13.14)$$

Hence using the linear approximation

$$f(x) \approx f(a) + f'(a)dx \quad (13.15)$$

$$f(4.2) \approx f(4) + f'(4)dx = 2 + (0.25)(.02) = 2.05 \quad (13.16)$$

To see how good the approximation is we square it:

$$2.05 \times 2.05 = 4.2025 \quad \square \quad (13.17)$$

Example 13.4 Find a linear approximation to \sqrt{x} near $x = 2$

From the previous example we use

$$f(x) = f(a) + f'(x)(x - a) = 2 + 0.25(x - 4) = 1 + .25x \quad (13.18)$$

In other words the line

$$y = 1 + .25x \quad (13.19)$$

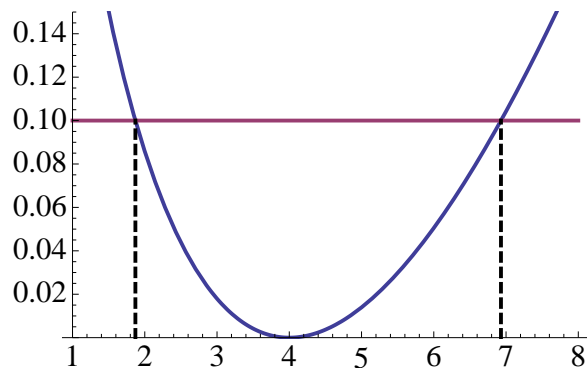
approximates the function $y = \sqrt{x}$ in some interval about $x = 4$.

Example 13.5 For what values of x is linear approximation $y = 1 + .25x$ to $y = \sqrt{x}$ accurate to within 0.1?

This question asks us to find the values of x for which

$$\left| \sqrt{x} - 1 - \frac{x}{4} \right| < 0.1 \tag{13.20}$$

From a graphing calculator we might estimate the range by plotting the left hand side of equation 13.20.



Alternatively, we can calculate the bounds using the quadratic equation.

$$-0.1 < \sqrt{x} - 1 - x/4 < 0.1 \tag{13.21}$$

This gives two conditions:

$$-.1 < \sqrt{x} - 1 - x/4 \text{ and } \sqrt{x} - 1 - x/4 < 0.1 \tag{13.22}$$

$$0 < \sqrt{x} - 0.9 - x/4 \text{ and } \sqrt{x} - 1.1 - x/4 < 0 \tag{13.23}$$

We can solve these using the quadratic formula if we substitute $u^2 = x$. Then

$$0 < u - 0.9 - u^2/4 \text{ and } u - 1.1 - u^2/4 < 0 \tag{13.24}$$

Multiplying each equation by -4 gives

$$0 > u^2 - 4u + 3.6 \text{ and } u^2 - 4u + 4.4 > 0 \quad (13.25)$$

The second equation is always true since

$$b^2 - 4ac = (-4)^2 - 4(1)(4.4) < 0 \quad (13.26)$$

and hence it never intersects the x -axis, yet is always positive (at $u = 0$, $u^2 - 4u + 4.4 = 4.4 > 0$). So any restriction is imposed by the first equation. Its endpoints occur at

$$u^2 - 4u + 3.6 = 0 \quad (13.27)$$

We have

$$\sqrt{b^2 - 4ac} = \sqrt{(-4)^2 - 4(3.6)} = \sqrt{1.6} \quad (13.28)$$

Hence

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{1.6}}{2} \approx \frac{4 \pm 1.26491}{2} = 1.36754, 2.63246 \quad (13.29)$$

Since $u = \sqrt{x}$ we get the endpoints in x as

$$x = 1.87108, 6.92982 \quad (13.30)$$

In other words the linear approximation is accurate to within 0.1 on the interval $[1.87108, 6.92982]$. \square

Example 13.6 Estimate $\sqrt[3]{25}$ using differentials.

Since

$$27 = 3^3 = 25 + 2 \quad (13.31)$$

we let $y = x^{1/3}$, $x = 27$, and $dx = -2$. Then

$$dy = f'(x)dx = \frac{1}{3}x^{-2/3}dx = -\frac{2}{3(\sqrt[3]{27})^2} = -\frac{2}{27} \approx 0.074 \quad (13.32)$$

hence

$$\sqrt[3]{25} \approx x + dx = 3 - 0.074 = 2.92593 \quad \square \quad (13.33)$$

Example 13.7. The area of a spherical soap bubble of radius r is $4\pi r^2$. Use differentials to estimate the change in the area as the radius increases from 1 cm to 1.1 cm.

We have

$$da = \left(\frac{d}{dr} 4\pi r^2 \right) dr = 8\pi r dr \quad (13.34)$$

If we let $r = 1$ and $dr = 1.1 - 1 = 0.1$, then da is the change in the area as the radius increases from 1 cm to 1.1 cm:

$$da = 8\pi(1)(.1) = 0.8\pi \approx 2.513 \quad (13.35)$$

The exact change is

$$\Delta a = 4\pi(1.1)^2 - 4\pi(1)^2 = .84\pi \approx 2.63894 \quad \square \quad (13.36)$$

Example 13.8 A tank is shaped like a cylinder with length L attached to hemispherical ends of radius r . Find a formula for the amount of paint in liters that is required to coat the tank, assuming that the thickness of the paint is 0.5 mm, $r = 1$ meter and $L=3$ meters (one cubic meter contains 1000 liters).

The volume of the tank is

$$V = (\text{cylinder of length } L \text{ and radius } r) + 2 \times (\text{end caps of radius } r) \quad (13.37)$$

$$= (\text{Volume of cylinder}) + (\text{Volume of Sphere}) \quad (13.38)$$

$$= \pi r^2 L + \frac{4}{3} \pi r^3 \quad (13.39)$$

Painting the tank increases the radius by dr without affecting the length. Hence we want to find out how dV depends on dr .

$$\frac{dV}{dr} = 2\pi r L + 4\pi r^2 \quad (13.40)$$

and therefore using $r = 1$, $L = 3$, and $dr = 0.001$,

$$dV = (2\pi r L + 4\pi r^2) dr \quad (13.41)$$

$$= 2\pi r(L + 2r) dr \quad (13.42)$$

$$= 2\pi(1)(3 + 2(1))(0.001) \quad (13.43)$$

$$= \frac{\pi}{100} \text{ cubic meters} \times \frac{1000 \text{ liters}}{\text{cubic meter}} \quad (13.44)$$

$$= 10\pi \text{ liters} \approx 31.4 \text{ liters} \quad \square \quad (13.45)$$

Topic 14

Extrema

Definition 14.1 A function $f(x)$ with a domain D , has an:

(a) **absolute maximum** (or **global maximum**) at c if for all x in D , $f, f(x) \leq f(c)$, and the number $f(c)$ is called the **maximum value** on D .

(b) **absolute minimum** (or **global minimum**) at c if for all x in D , $f, f(x) \geq f(c)$, and the number $f(c)$ is called the **minimum value** on D .

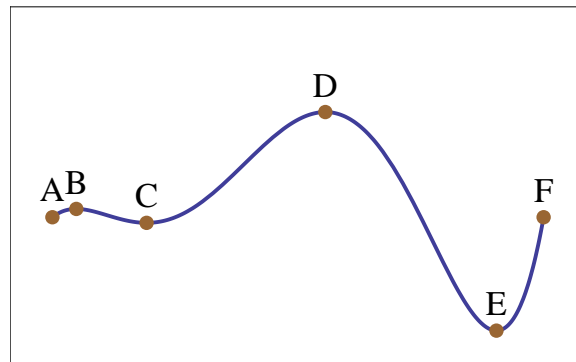
(c) **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ for all x **near** c .

(d) **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ for all x **near** c .

Remark 14.2 Every global maximum (minimum) is also a local maximum (minimum).

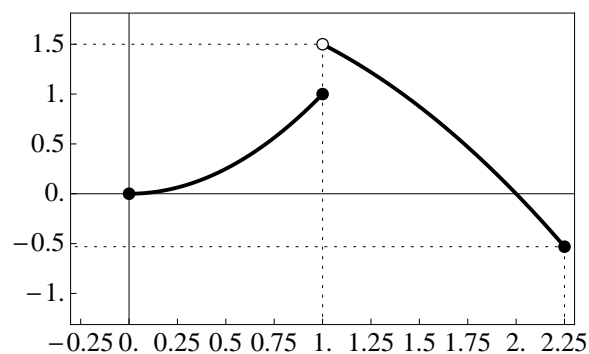
Normally, however, we take the above remark as obvious. Hence whenever we are listing out the absolute and relative extrema, we identify points either as absolute or relative, and don't explicitly state that the absolute extrema are also relative.

In the following figure, D is the absolute maximum; E is the absolute minimum; A and C are relative minima; and B and F are relative maxima.



Theorem 14.3 Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$ then f takes on both an absolute maximum value $f(c)$ at some point $c \in [a, b]$, and an absolute minimum value $f(d)$ at some point $d \in [a, b]$.

The extreme value theorem only applies on a closed interval (not an open interval); and if the function is not continuous, then it is possible that there are no extrema. For example, in the following figure, the function does not have any maximum value on $[0, 2.25]$ even though it is defined for the entire interval, because it is not continuous at $x = 1$. Similarly, there is no maximum value on $(1, 2.25]$, even though the function is continuous on the entire interval, because the interval is an open interval.



Theorem 14.4 Fermat's Theorem If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem makes it easy to calculate the extrema of continuous functions.

Definition 14.5 A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Generally, the critical numbers occur where: (a) $f'(x) = 0$; (b) endpoints; (c) points of discontinuity; (d) corners (places where f' is not continuous).

Theorem 14.6 Local maxima and minima of f occur at critical numbers.

Procedure: The Closed Interval Method. To find the absolute maximum and minimum of a function on a closed interval:

1. Find the values of f at the endpoints of the interval.
2. Find the values of f at all points where $f'(x) = 0$.
3. Find the values of f at all points where $f'(x)$ does not exist, such as corners.
4. The absolute maximum (minimum) is the largest (smallest) value of the numbers calculated in steps 1-3.

Example 14.1 Find the critical numbers of $f(t) = 3t^4 + 4t^3 - 6t^2$.

Since the domain is not given, we must infer it from the function, and since a polynomial is defined on all x , the domain is $(-\infty, \infty)$. Hence there are no endpoints.

Since a polynomial is everywhere differentiable, there are no points where the derivative is discontinuous (i.e., no corners).

The remaining critical points occur when $f'(x) = 0$. So we solve

$$0 = f'(x) = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1) = 12t(t - r_1)(t - r_2) \quad (14.1)$$

where r_1, r_2 are the roots of the quadratic equation

$$0 = t^2 + t - 1 \quad (14.2)$$

Hence ($a = 1; b = 1; c = -1$),

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{5}}{2} = -1 \pm \frac{\sqrt{5}}{2} \quad (14.3)$$

Therefore the critical points occur at

$$x = 0, -1 - \frac{\sqrt{5}}{2}, -1 + \frac{\sqrt{5}}{2} \quad \square \quad (14.4)$$

Example 14.2 Find the absolute maximum and minimum values of

$$f(x) = x^3 - 6x^2 + 9x + 2 \quad (14.5)$$

on the interval $[-1, 4]$

(1) at the endpoints,

$$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -14 \quad (14.6)$$

$$f(4) = 4^3 - 6(4)^2 + 9(4) + 2 = 64 - 96 + 36 + 2 = 6 \quad (14.7)$$

(2) There are no points of discontinuity or corners because f is a polynomial which is continuously differentiable everywhere.

(3) The remaining critical points are when $f'(x) = 0$. So

$$0 = f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3) \quad (14.8)$$

which gives additional critical points at $x = 1$ and $x = 3$. But

$$f(1) = 1^3 - 6(1) + 9(1) + 2 = 6 \quad (14.9)$$

$$f(3) = 3^3 - 6(3^2) + 9(3) + 2 = 27 - 54 + 27 + 2 = 2 \quad (14.10)$$

In summary, here are the critical points:

x	$f(x)$	type of critical point
-1	-14	left end point
1	6	$f'(x) = 0$
3	2	$f'(x) = 0$
4	6	right endpoint

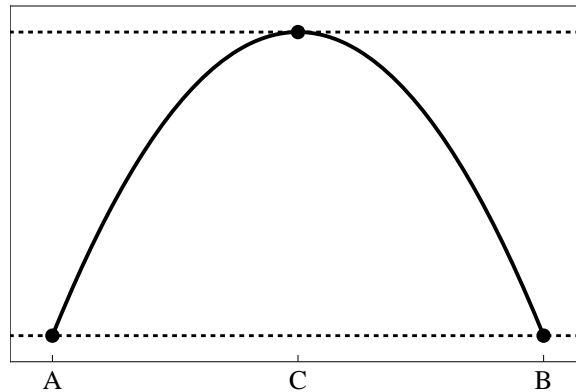
We see that $x = 1$ and $x = 4$ tie for the point where the absolute maximum occurs; and the absolute minimum occurs at $x = -1$. The absolute maximum value of the function on $[-1, 4]$ is 6, and the absolute minimum value of the function on $[-1, 4]$ is -14. \square

Topic 15

Mean Value Theorem

Theorem 15.1 Rolle's Theorem Let f be function that is (a) continuous on $[a, b]$; (b) differentiable on (a, b) ; such that $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

In the following figure, the tangent line at C is horizontal.

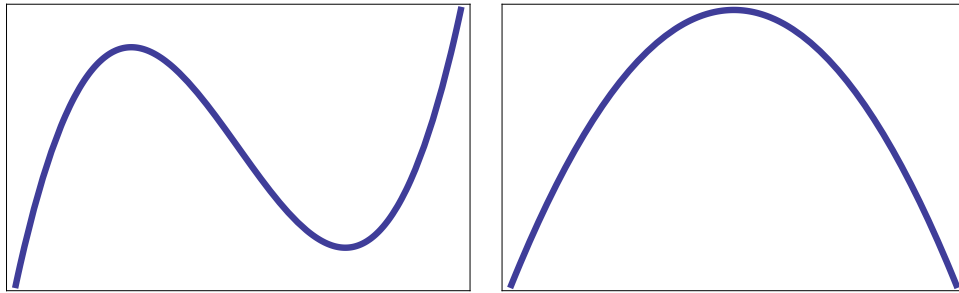


Proof. We will consider three cases: (a) $f(x)$ is a constant; (b) there is some number $x \in (a, b)$ such that $f(x) > f(a)$; and (c) there is some number $x \in (a, b)$ such that $f(x) < f(a)$.

Case (a). $f(x) = K$ for some constant K . Then every point $x \in (a, b)$ satisfies

$f'(x) = 0$ hence we can pick any point as C .

Case(b). There is some number $x \in (a, b)$ such that $f(x) > f(a)$. The picture looks like one of these figures:



By the extreme value theorem, f has an absolute maximum on $[a, b]$. Call this point c .

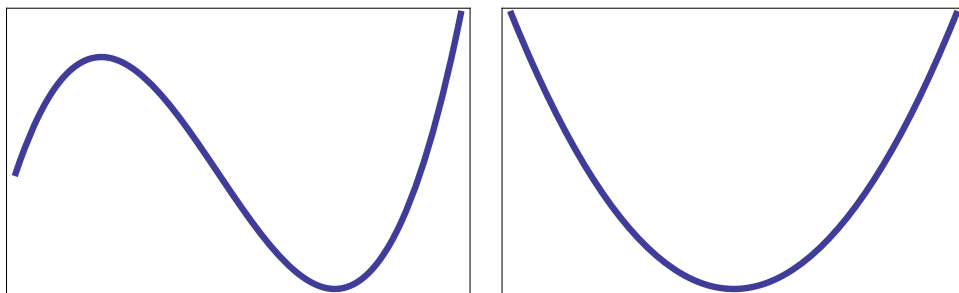
Since there is some point x with $f(x) > f(a)$, then $f(c)$ is at least as big as $f(x)$. Hence $c \neq a$.

Since $f(a) = f(b)$, we also know $f(x) > f(b)$. Hence $c \neq b$.

Hence $c \in (a, b)$. Hence there is a local maximum at c (as well as a global maximum).

By Fermat's theorem, $f'(c) = 0$.

Case(c). There is some number $x \in (a, b)$ such that $f(x) < f(a)$. The picture looks like one of these figures:



The proof is identical to case (b) with the $>$ replaced by $<$. \square

Example 15.1 Prove that $f(x) = x^3 + x$ has exactly one real root.

To show that at least one real root exists we calculate

$$f(-1) = (-1)^3 - 1 = -2 \quad (15.1)$$

$$f(1) = (1)^3 + 1 = 2 \quad (15.2)$$

By the intermediate value theorem, since f is continuous on $[-1, 1]$ and $-1 < 0 < 1$, then there is some point $c \in [-1, 1]$ such that

$$f(c) = 0 \quad (15.3)$$

Thus there exists one root that is between -1 and 1.

To prove that that is the only root, suppose that there are two roots. Call these roots a and b . Then, since they are roots,

$$f(a) = f(b) \quad (15.4)$$

Since f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. then f satisfies the conditions of Rolle's theorem. By Rolle's theorem, there is some point $d \in (-1, 1)$ such that $f'(d) = 0$. But

$$f'(x) = 3x^2 + 1 > 0 \quad \forall x \quad (15.5)$$

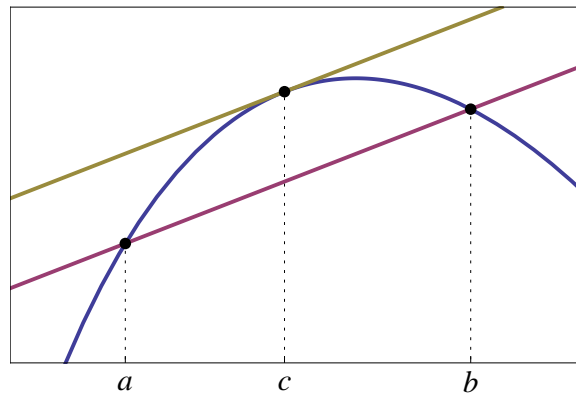
so it is impossible to have $f'(d) = 0$. Thus there is a contradiction in the proof, so the assumption we made must be wrong. Hence there cannot be more than one root.

\square

Theorem 15.2 Mean Value Theorem Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . Then there is a real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (15.6)$$

The mean value theorem tells us that there is a point between a and b that has the same slope as the secant line joining $(a, f(a))$ and $(b, f(b))$.



Proof. We first calculate the equation of the secant line between $(a, f(a))$ and $(b, f(b))$.

Then slope is

$$m = \frac{f(b) - f(a)}{b - a} \quad (15.7)$$

hence the line satisfies the equation

$$y = f(a) + m(x - a) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \quad (15.8)$$

We will define a new function $g(x)$ as the difference:

$$g(x) = f(x) - y = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \quad (15.9)$$

Then by substitution we see that

$$g(b) = g(a) = 0 \tag{15.10}$$

Since f is continuous on $[a, b]$ and differentiable on (a, b) , so is g ; hence g satisfies the conditions of Rolle's theorem.

Hence there is a point c between a and b such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \tag{15.11}$$

Setting $0 = g'(c)$ gives

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \tag{15.12}$$

and therefore

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{15.13}$$

□

Theorem 15.3 If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on (a, b) .

Proof. Let $u, v \in (a, b)$ be any two numbers such that $u < v$. Since

$$a < u < v < b \tag{15.14}$$

and f is differentiable on (a, b) then it is differentiable on (u, v) , because $(u, v) \subset (a, b)$.

Similarly, since f is continuous on $[a, b]$, it is also continuous on (u, v) .

Hence f satisfies the mean value theorem on (u, v) .

Hence there is some number $c \in (u, v)$ such that

$$f'(c) = \frac{f(u) - f(v)}{u - v} \quad (15.15)$$

and therefore

$$f(u) - f(v) = f'(c)(u - v) = 0 \quad (15.16)$$

where the last equality follows because $f'(x) = 0$ for all $x \in (a, b)$. Hence

$$f(u) = f(v) \quad (15.17)$$

Thus we can pick any two points in (a, b) and they are equal. Hence all points in (a, b) are equal. Hence the function is constant. \square

Corollary 15.4 If $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f - g$ is a constant on (a, b) , i.e.,

$$f(x) = g(x) + C \quad (15.18)$$

for some constant C .

Proof. Define

$$F(x) = f(x) - g(x) \quad (15.19)$$

Then

$$F'(x) = f'(x) - g'(x) = 0 \quad (15.20)$$

for all x in (a, b) . Hence by the previous theorem $F(x)$ is constant. \square

Example 15.2 Verify the mean value theorem for

$$f(x) = \frac{x}{x+2} \text{ on } [1, 4] \quad (15.21)$$

and find all numbers c that satisfy the conclusion of the Mean Value Theorem.

We have $a = 1$ and $b = 4$.

The function $f(x)$ is a rational function, which is continuous and differentiable everywhere except where the denominator is zero. This only occurs when $x = -2$. Since -2 is between 1 and 4, the function is continuous on $[1, 4]$ and differentiable on $(1, 4)$.

Hence the mean value theorem applies to tell us that there is a point $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{4/6 - 1/3}{4 - 1} = \frac{1}{9} \quad (15.22)$$

By the quotient rule,

$$f'(x) = \frac{(x+2)(1) - (x)(1)}{(x+2)^2} = \frac{2}{(x+2)^2} \quad (15.23)$$

Hence the point that satisfies the mean value theorem is the solution of

$$\frac{1}{9} = \frac{2}{(x+2)^2} \quad (15.24)$$

$$(x+2)^2 = 18 \quad (15.25)$$

$$(x+2) = \pm\sqrt{18} = \pm 3\sqrt{2} \quad (15.26)$$

$$x = -2 \pm 3\sqrt{2} \quad (15.27)$$

The solution with the negative sign is not in the interval $(1, 4)$. Hence the point that

satisfies the mean value theorem is

$$x = -2 + 3\sqrt{2} \quad \square \quad (15.28)$$

Example 15.3 Determine if the mean value theorem applies to

$$f(x) = x^3 + x - 4 \quad (15.29)$$

on $[-1,2]$, and if so find the point c predicted by the theorem.

The function is a polynomial, so it is continuous and differentiable for all x . Hence f is continuous on $[-1,2]$ and differentiable on $(-1, 2)$. Hence the mean value theorem applies.

The mean value theorem says that for some c in $[-1, 2]$,

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{(8 + 2 - 4) - (-1 - 1 - 4)}{2 - (-1)} = 4 \quad (15.30)$$

Since

$$f'(x) = 3x^2 + 1 \quad (15.31)$$

then the number c satisfies

$$4 = 3c^2 + 1 \implies 3 = 3c^2 \implies c = \pm 1 \quad (15.32)$$

But -1 is not in the interval $(-1, 2)$ hence the only point predicted by the mean value theorem is $c = 1$. \square

Topic 16

Curve Sketching

The derivative gives us a lot of information about the shape of the curve. In fact, just by looking at the values of the first and second derivatives at certain points, we are often able to sketch plots of entire curves.

The derivative gives us the slope of the function. So where the derivative is positive, the slope is positive, and where the derivative is negative, the slope is negative. This gives us the following rule:

Increasing/Decreasing Test:

- (a) If $f'(x) > 0$ the function is increasing;
- (b) If $f'(x) < 0$, the function is decreasing;
- (c) If $f'(x) = 0$, the function has a horizontal tangent.

Let us consider the following example:

$$y = 4x^3 - 12x^2 + 5 \tag{16.1}$$

Differentiating,

$$y' = 12x^2 - 24x \tag{16.2}$$

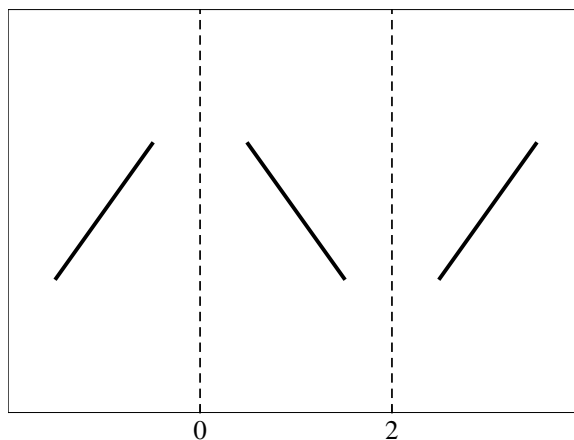
We are interested in where y' is positive and where it is negative. At the transition points between intervals we have

$$0 = 12x^3 - 24x = 12x(x - 2) \quad (16.3)$$

The transition points are at $x = 0$ and $x = 2$. So we have three intervals to consider: $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$. We can determine what is going on in each interval by considering a point inside each interval.

Interval	Point (x)	$f'(x)$	Slope
$(-\infty, 0)$	-1	36	increasing
$(0, 2)$	1	-12	decreasing
$(2, \infty)$	3	36	increasing

So we can begin to make a sketch. So far, we have something like this:



The diagonal bars tell us the general direction of increase or decrease. But we have not yet looked at enough information to figure out where the values lie. To do that, we want to find the extrema.

From the previous section, we know that the extrema lie at critical points. There are

no endpoints or points of discontinuity on this function, so we only have to look at the points where $f'(x) = 0$; we have already determined that for this function, this occurs at $x = 0$ and $x = 2$.

Since f is increasing to the left of $x = 0$ and decreasing to the right, we must have a local maximum at $x = 0$.

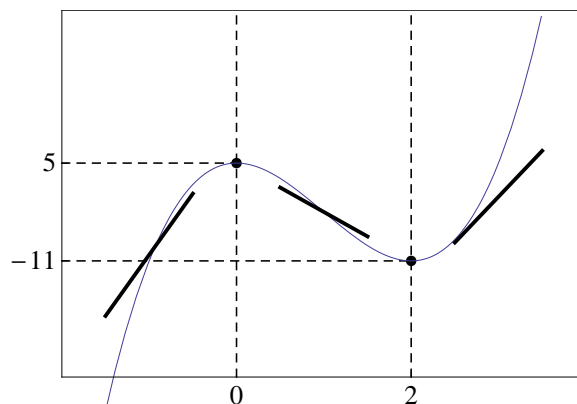
Similarly, since f is decreasing to the left of $x = 2$ and increasing to the right of $x = 2$, we must have a local minimum at $x = 2$.

To get y -values we plug back into f .

$$f(0) = 5; f(2) = 4(2^3) - 12(2^2) + 5 = -11 \quad (16.4)$$

Thus $(0, 5)$ is a local maximum and $(2, -11)$ is a local minimum.

With this we can expand our plot. The correct answer is also plotted.



In this example, the intervals between the critical points alternated increasing, decreasing, increasing. However, this does not always happen. Consider, for example,

$$y = x^3 \quad (16.5)$$

which has a critical point at $x = 0$ but is **always increasing** except at $x = 0$.

We have used the following rule to determine where the function turns from increasing to decreasing or vice-versa, or neither.

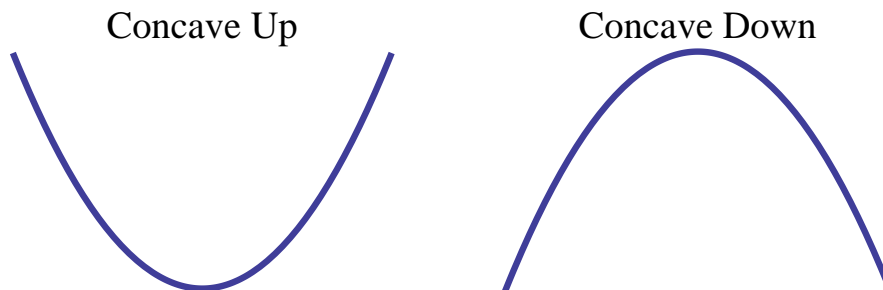
The First Derivative Test: Let c be a critical number of f . Then

- (a) If f' changes from positive to negative at c , then f has a local maximum at c .
- (b) If f' changes from negative to positive at c then f has a local minimum at c .
- (c) If f' does not change sign from one side of c to the other (even though we may have $f'(c) = 0$), then there is not an extrema at c .

The **second derivative** also tells us something about the shape of f .

By the increasing/decreasing test, when $f'' > 0$, then f' is increasing. Thus the slope is getting larger (or less negative) at a faster and faster rate. We call this **concave upward**.

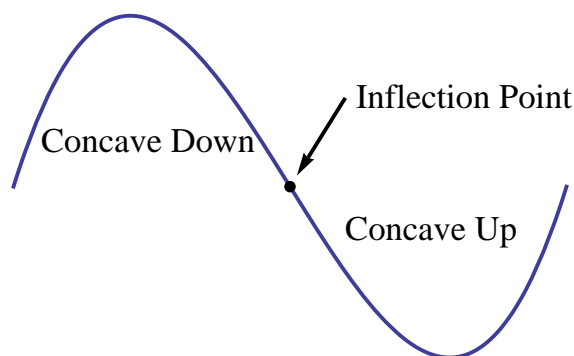
Similarly, when $f'' < 0$, then f' is decreasing. Thus f' is getting smaller (or less large, if $f' > 0$) at a faster and faster rate. We call this **concave downward**.



Concavity Test: Let f be a function defined on some interval I

- (a) If $f''(x) > 0$ for all $x \in I$ then f is concave up on I ;
- (b) If $f''(x) < 0$ for all $x \in I$ then f is concave down on I

A point where the concavity changes - either from concave up to concave down or concave down to concave up, is called an **inflection point**. At an inflection point $f''(c) = 0$, the signs of $f''(x)$ are different on either side of c .



Since a concave upwards function is shaped like a bowl, the bottom of the bowl is a local minimum. Similarly, since a concave downwards function is shaped like a hill, the top of the hill is a local maximum. This leads to the following **Second Derivative Test** for an extremum.

Second Derivative Test: Let f be a function that is continuous near c . Then

- (a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c .
- (b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c .

Example 16.1 Sketch the curve

$$y = x^4 - 4x^3 \tag{16.6}$$

using the various rules of sketching we have described.

Calculating the first two derivatives we have

$$y' = 4x^3 - 12x^2 = 4x^2(x - 3) \tag{16.7}$$

$$y'' = 12x^2 - 24x = 12x(x - 2) \tag{16.8}$$

From the first equation we find that there are critical points at $x = 0$ and $x = 3$.

From the second equation we find that

$$f''(0) = 0 \implies \text{Neither a maximum nor a minimum} \quad (16.9)$$

$$f''(3) = 12(3)(3 - 2) > 0 \implies \text{concave up} \implies \text{local minimum} \quad (16.10)$$

The critical points divide the domain into three regions:

$$x < 0, \quad 0 < x < 3, \quad 3 < x \quad (16.11)$$

We can use the first derivative test to determine the direction of the slope in each region. We pick points $x = -1, x = 1, x = 4$. Then

$$f'(-1) = (4)(-1)^2(-1 - 3) < 0 \implies \text{decreasing} \quad (16.12)$$

$$f'(1) = 4(1)^2(1 - 3) < 0 \implies \text{decreasing} \quad (16.13)$$

$$f'(4) = 4(4)^2(4 - 3) > 0 \implies \text{increasing} \quad (16.14)$$

Since the slope is decreasing on the first two intervals, the point $x = 0$, where $f''(0) = 0$, is an inflection point.

We already know from the second derivative test that $x = 3$ is a local minimum; we can also verify this with the first derivative test, since $y' < 0$ for $x < 3$ and $y' > 0$ —for $x > 3$.

The second derivative can tell us one more piece of information. We see that $y'' = 0$ when either $x = 0$ or $x = 2$. We already know that $x = 0$ is an inflection point; so is $x = 2$.

Finally, we can calculate the function at the inflection points and at the minimum.

$$f(0) = 0 \implies (0, 0) \text{ is an inflection point} \quad (16.15)$$

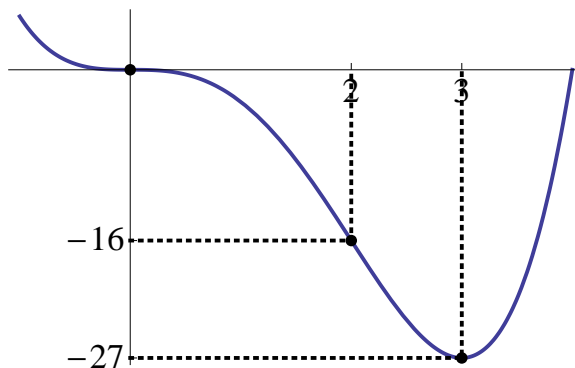
$$f(2) = 2^4 - 4(2^3) = 16 - 32 = -16 \implies (2, -16) \text{ is an inflection point} \quad (16.16)$$

$$f(3) = 3^4 - 4(3^3) = 81 - 4(27) = -27 \implies (3, -27) \text{ is a local minimum} \quad (16.17)$$

We can collect all this information together in the following table.

	f'		f''	
$x < 0$	-	Decreasing	+	Concave Up
$x = 0$	0	Critical Point	0	Inflection Point (Concavity Changes)
$0 < x < 2$	-	Decreasing	-	Concave Down
$x = 2$	-	Decreasing	0	Inflection Point
$2 < x < 3$	-	Decreasing	+	Concave Up
$x = 3$	0	Critical Point	+	Concave Up: Local Minimum (2nd Der Test)
$x > 3$	+	Increasing	+	Concave Up

From this information we can draw a sketch.



Example 16.2 Find the point where

$$f(x) = \frac{x + 1}{\sqrt{x^2 + 1}} \quad (16.18)$$

increases most rapidly.

Topic 17

Horizontal Asymptotes

Definition 17.1 Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L \tag{17.1}$$

means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large, and the line $y = L$ is called a **horizontal asymptote** of f .

If f is defined on $(-\infty, a)$ then

$$\lim_{x \rightarrow -\infty} f(x) = L \tag{17.2}$$

means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative, and the line $y = L$ is also called a **horizontal asymptote** of f .

Curves with horizontal asymptotes usually appear to give

$$f(x) \rightarrow \frac{\infty}{\infty} \tag{17.3}$$

under the substitution rule. In this case we need to do additional algebra to determine the actual limit. In the limit we can generally use the approximations

$$\frac{1}{\infty} \rightarrow 0 \quad \text{and} \quad \frac{1}{0} \rightarrow \infty \quad (17.4)$$

Our aim is then to convert forms that contain things like

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty} \quad (17.5)$$

to something that we can evaluate.

Example 17.1 Sketch the plot of

$$f(x) = \frac{3x + 4}{5x - 7} \quad (17.6)$$

Since f is a rational function, the natural domain of $f(x)$ is set of all points except where the denominator is zero. Hence we accept all x except for $x = 7/5 = 1.4$.

The line $x = 1.4$ is a vertical asymptote of $f(x)$.

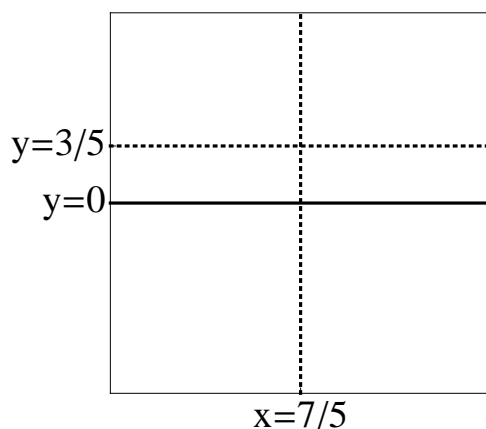
We next observe that

$$\frac{3x + 4}{5x - 7} \rightarrow \frac{\infty}{\infty} \quad (17.7)$$

as $x \rightarrow \infty$, so we use the following trick of multiplication by 1 in the form of $(1/x)/(1/x)$:

$$\frac{3x + 4}{5x - 7} = \frac{\frac{3x}{x} + \frac{4}{x}}{\frac{5x}{x} - \frac{7}{x}} = \frac{3 + \frac{4}{x}}{5 - \frac{7}{x}} \rightarrow \frac{3 + \frac{4}{\infty}}{5 - \frac{7}{\infty}} \rightarrow \frac{3 + 0}{5 + 0} = \frac{3}{5} \quad (17.8)$$

Hence $y = 3/5$ is a **horizontal** asymptote.



Before we can sketch the curve we need to determine which side of the curve the function lies on. The function is continuous everywhere except at the **horizontal** asymptote $x = 1.4$, so it has two parts.

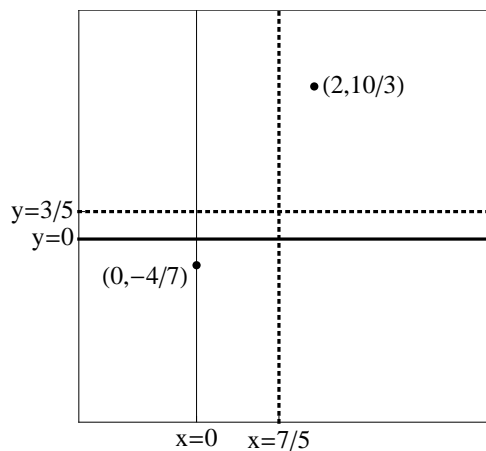
To find out whether these parts lie above or below the **horizontal** asymptote, we can plug in points. We observe that

$$f(0) = -\frac{4}{7} \quad (17.9)$$

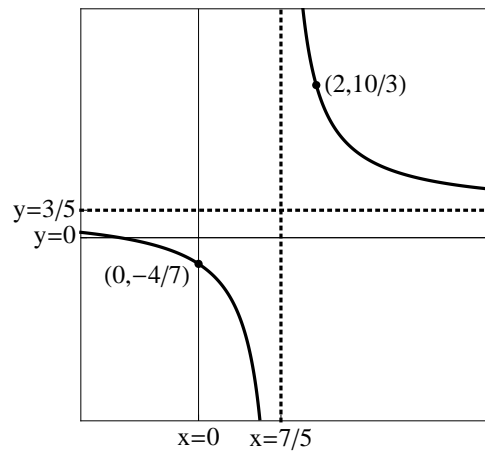
and

$$f(2) = \frac{3(2) + 4}{5(2) - 7} = \frac{10}{3} \quad (17.10)$$

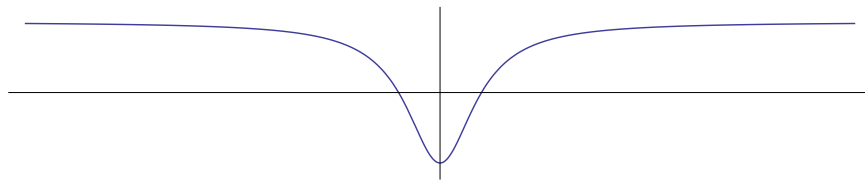
This gives us the two points $(0, -4/7)$ and $(2, 10/3)$ and hence we have the following:



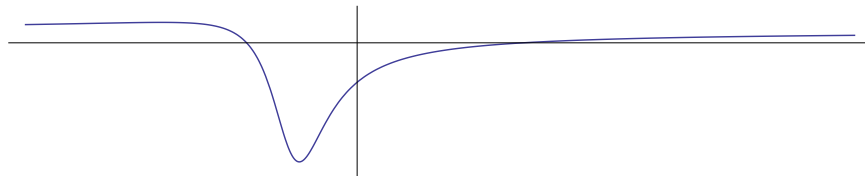
Knowing the asymptotes now gives us the ability to sketch the rest of the curve.



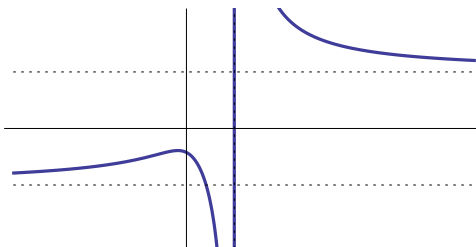
Example 17.2 $f(x) = \frac{x^2 - 1}{x^2 + 1}$



Example 17.3 $f(x) = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$



Example 17.4 $f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$



We can also get infinity limits at $\pm\infty$.

Example 17.5 $\lim_{x \rightarrow \infty} x^3 + x$

Example 17.6 $\lim_{x \rightarrow \infty} \frac{x^2 + x}{3 - x}$

Example 17.7 Sketch $y = \frac{1 + 2x^2}{1 + x^2}$

Horizontal Asymptotes: $y = 2$

Vertical Asymptotes: None

To see if the function ever crosses the horizontal asymptote: Solve $y = 2$ for x .

$$2 = y \implies 2 = \frac{1 + 2x^2}{1 + x^2} \implies 2 + 2x^2 = 1 + 2x^2 \implies 1 = 2 \implies \text{No solution}$$

Hence the function never crosses the asymptote.

$$\text{Derivatives: } y' = \frac{2x}{(1 + x^2)^2}, \quad y'' = \frac{2(1 - 3x^2)}{(1 + x^2)^3}$$

Critical Points: y and y' are defined everywhere so the only critical points occur when $y' = 0$. This only occurs at $x = 0$.

To find out where the minimum is on the plot we find $f(0) = 1$, so the point is $(0, 1)$.

Intervals where the function is increasing decreasing: $y' > 0$ when $x > 0$ and $y' < 0$ when $x < 0$. So the function is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$

Second Derivative Test at Critical Points where $y' = 0$: $y''(0) = 2 > 0$ so the function is concave up at $x = 0$, and a local minimum occurs at $x = 0$.

Intervals of Concavity: Candidate inflection points occur when $3x^2 = 1$ or $x = \pm\sqrt{3}$.

This gives three intervals:

$$(-\infty, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}), (\sqrt{3}, \infty)$$

We already know that the middle interval is concave up by the second derivative test at the critical point at $x = 0$:

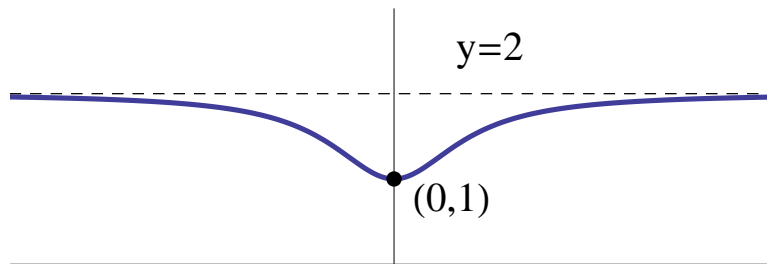
$$f''(0) = 2 > 0 \implies \text{concave up on } (-\sqrt{3}, \sqrt{3})$$

To test the other intervals we pick points, say $x = 2$ and $x = -2$.

$$f''(-2) = \frac{-22}{125} < 0 \implies \text{concave down on } (-\infty, -\sqrt{3})$$

$$f''(2) = \frac{-22}{125} < 0 \implies \text{concave down on } (\sqrt{3}, \infty)$$

Inflection points: at $x = -\sqrt{3}$ and $x = \sqrt{3}$.



Example 17.8 Sketch $y = \frac{1 + 2x^2}{1 - x^2}$

The only difference between this function and the previous example is the minus sign in the denominator.

Horizontal Asymptote: $y = -2$

Vertical Asymptote: $x = \pm 1$.

$$\text{Derivatives: } y' = \frac{6x}{(x^2 - 1)^2}, y'' = -\frac{6(1 + 3x^2)}{(x^2 - 1)^3}$$

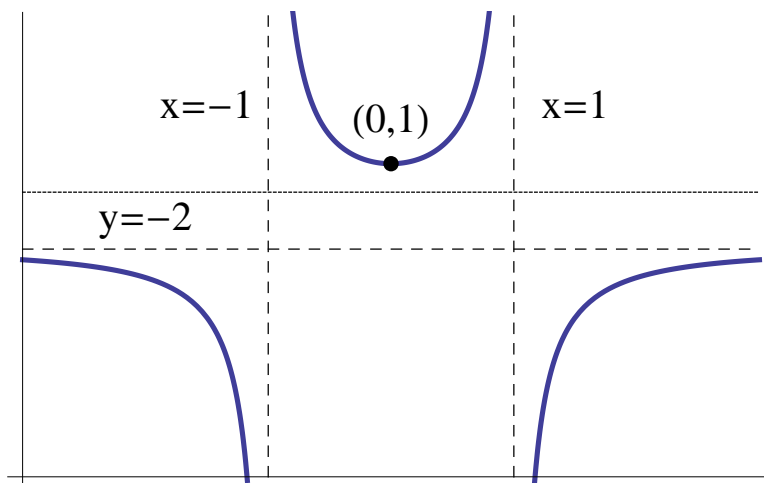
Critical Point: $x' = 0 \implies x = 0$. At $x = 0$, $y = 1$.

Second Der. Test: At $x = 0$, $y'' = 6$ so the function is concave up (a local min.).

Intervals of Concavity: Test $y'' = 0$. This does not occur so there are no inflection points. But there are vertical asymptotes so the concavity could change there:

$$(-\infty, -1), (-1, 1), (1, \infty)$$

Test points in the other intervals: $y''(2) = -26/9$ and $y''(-2) = -26/9$ so the outer intervals are both concave down.



Definition 17.2 Precise Definition of a Limit at ∞ . Let f be defined on an interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L \tag{17.11}$$

means that for every $\epsilon > 0$ there exists some number N (large) such that whenever $x > N$, we have

$$|f(x) - L| < \epsilon \tag{17.12}$$

Definition 17.3 Precise Definition of a Limit at $-\infty$. Let f be defined on an interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L \tag{17.13}$$

means that for every $\epsilon > 0$ there exists some number N (large, negative) such that whenever $x < N$, we have

$$|f(x) - L| < \epsilon \tag{17.14}$$

Definition 17.4 Precise Definition of an Infinite Limit . Let f be defined on

an interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad (17.15)$$

means that for every (large) $M > 0$ there exists some (large) number N such that whenever $x > N$, we have

$$f(x) > M \quad (17.16)$$

A similar definition can be made for the limit being equal to $-\infty$.

Summary of Curve Sketching

1. Determine the **natural domain** of the function.
2. Find all the **intercepts**.
3. Look for **symmetry**: is the function odd or even?
4. Look for **asymptotes**: horizontal, vertical, and slanted.
5. Determine points where $f' = 0$ and $f'' = 0$.
6. Identify regions of **increase** and **decrease** from sign of f' .
7. Identify regions of **concavity** and **inflection** points from f'' .
8. Identify all local **extrema** from the first and/or second derivative tests.
9. Sketch!

Topic 18

Optimization

In this section we will complete several examples that illustrate solving optimization (maximize/minimize) problems with calculus. Optimization is the process of finding the absolute maximum or absolute minimum of a function over a specific domain.

First Derivative Test for Absolute Extreme Values. Let c be a critical point of a continuous function defined on some interval. Then $f(c)$ is

- (a) An **absolute maximum** if $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$;
- (b) An **absolute minimum** if $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$.

Second Derivative Test for Absolute Extreme Values. Let c be a critical point of a continuous function defined on some interval with $f'(c) = 0$. Then $f(c)$ is

- (a) An **absolute maximum** if $f''(x) < 0$ for all x ;
- (b) An **absolute minimum** if $f''(x) > 0$ for all x .

Example 18.1 A farmer has 3000 feet of fence and wants to fence off three sides of a field that borders a river, There is no fence along the river. What are the dimensions of the largest rectangular field that she can fence off?

Define the variables:

$$x = \text{length of side of field parallel to the river} \quad (18.1)$$

$$y = \text{length of side of field perpendicular to the river} \quad (18.2)$$

$$A = xy = \text{Area} \quad (18.3)$$

$$P = x + 2y = \text{Amount of Fence Used} = 3000 \quad (18.4)$$

Substituting

$$x = 3000 - 2y \quad (18.5)$$

into equation 18.3 gives

$$A = xy = (3000 - 2y)y = 3000y - 2y^2 \quad (18.6)$$

The maximum area occurs when $A(y)$ is maximized over its allowed domain.

But this means the area must be positive. The area is zero when either $y = 0$ or $y = 1500$; $A(y) > 0$ on $(0, 1500)$ and is negative otherwise. Hence we must restrict y to $0 \leq y \leq 1500$.

Differentiating the equation for $A(y)$,

$$A' = 3000 - 4y \quad (18.7)$$

Critical points occur when $A' = 0$ which implies

$$A' = 0 \implies 0 = 3000 - 4y \implies y = 3000/4 = 750 \quad (18.8)$$

At $y = 750$ we get

$$A(750) = (30002(750))(750) = 1,125,000 \quad (18.9)$$

Differentiating a second time we get

$$A'' = -2y - 2(750) < 0 \implies \text{local maximum} \quad (18.10)$$

Hence $y = 750$ gives a local maximum. To get the other dimension we use equation 18.5 to get

$$x = 3000 - 2(750) = 3000 - 2(750) = 1500 \quad (18.11)$$

hence the largest field that can be fenced in this manner is 1500×750 feet, with the 1500 feet parallel to the river. \square

Example 18.2 Find two positive numbers whose product is 100 and whose sum is a minimum.

We want to find x and y such that

$$xy = 100 \quad (18.12)$$

and

$$S(x) = x + y = x + \frac{100}{x} \quad (18.13)$$

is minimized. Setting $S'(x) = 0$ gives

$$0 = S'(x) = 1 - \frac{100}{x^2} \implies 100 = x^2 \implies x = 10 \quad (18.14)$$

Hence $x = 10$ and $y = 100/10 = 10$.

To verify that this is a minimum, and not a maximum, we find

$$S''(x) = \frac{200}{x^3} \implies S''(10) = \frac{200}{1000} > 0 \implies \text{local minimum} \quad \square \quad (18.15)$$

Example 18.3 Suppose you want to make a rectangular one liter box with an open top and square bottom (1000 cm^3) that uses the least amount of material.

Let the base of the box have dimensions $x \times x$, and let the height be h .

The volume of the box is

$$1000 = x^2 h \implies h = \frac{1000}{x^2} \quad (18.16)$$

The material used is

$$M(xh) = x^2 + 4xh = x^2 + (4x) \left(\frac{1000}{x^2} \right) = x^2 + \frac{4000}{x} \quad (18.17)$$

Setting $M'(x) = 0$ gives

$$0 = 2x - \frac{4000}{x^2} \implies 2x^3 = 4000 \implies x = (2000)^{1/3} \approx 12.5992 \quad (18.18)$$

From equation [18.16](#)

$$h = \frac{1000}{2000^{2/3}} \approx 6.29961 \quad (18.19)$$

hence the dimensions are approximately $6.29961 \text{ cm} \times 12.5992 \text{ cm} \times 12.5992 \text{ cm}$.

□

Example 18.4 Find the point on the line $6x + y = 9$ that is closet to the origin, and find the distance.

The distance from the origin to a point (x, y) is

$$d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (9 - 6x)^2} \quad (18.20)$$

Since d^2 is also minimized when d is minimized, we find

$$f(x) = d^2 = x^2 + (9 - 6x)^2 \quad (18.21)$$

The critical point is at $f'(x) = 0$,

$$0 = f'(x) = 2x + 2(9 - 6x)(-6) = 2x - 108 + 72x = 74x - 108 \quad (18.22)$$

$$\implies x = \frac{108}{74} = \frac{54}{37} \quad (18.23)$$

To verify that this is a minimum,

$$f''(x) = 74 > 0 \implies \text{concave up} \implies \text{minimum} \quad (18.24)$$

Hence

$$y = 9 - 6x = 9 - 6(54/37) = 9/37 \quad (18.25)$$

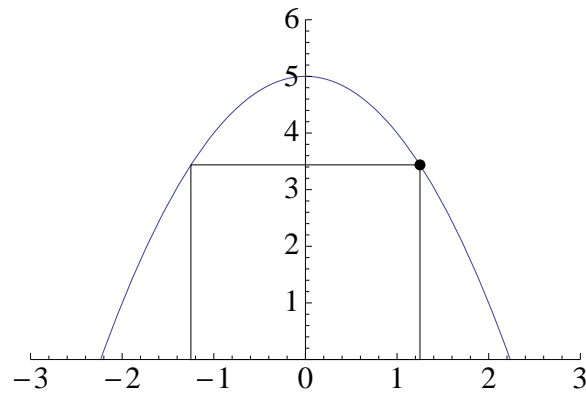
So the point nearest the origin is $(54/37, 9/37)$ and the distance is

$$d = \sqrt{x^2 + y^2} = \frac{9}{\sqrt{37}} \quad \square \quad (18.26)$$

Example 18.5 A rectangle is inscribed with its base on the x-axis and its upper

corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

Let (x, y) denote the point on the upper right-hand corner of the rectangle as illustrated below:



Then the area of the rectangle is

$$A = 2xy \tag{18.27}$$

We want to maximize A subject to $y = 5 - x^2$. Substituting for y ,

$$A = 2x(5 - x^2) = 10x - 2x^3 \tag{18.28}$$

Differentiating,

$$A' = 10 - 6x^2 \tag{18.29}$$

$$A'' = -12x \tag{18.30}$$

The critical points occur when

$$0 = A' = 10 - 6x^2 \implies x^2 = \frac{10}{6} = \frac{5}{3} \implies x = \pm\sqrt{\frac{5}{3}} \tag{18.31}$$

We choose the positive square root to ensure that $A''(x) < 0$ (hence the function is

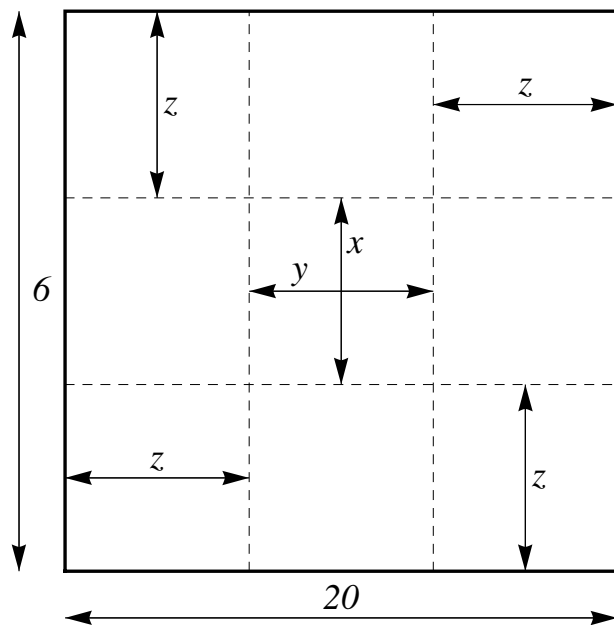
concave down and at a local maximum). Hence

$$y = 5 - x^2 = 5 - \frac{5}{3} = \frac{10}{3} \quad (18.32)$$

$$A = 2xy = 2 \left(\frac{10}{3} \right) \left(\sqrt{\frac{5}{3}} \right) = \frac{20\sqrt{5}}{3\sqrt{3}} = \frac{20\sqrt{15}}{9} \quad (18.33)$$

Example 18.6 A box is made out of a piece of cardboard that has dimensions 6×20 . Squares of equal size will be cut out of each corner and then the ends and sides folded up to form a box with an open top. Find the dimensions of the resulting box that has the largest volume.

Let the box have a base of dimensions $x \times y$, and height z . Then the four squares on the corners must have dimensions $z \times z$, as illustrated below.



We need to maximize the volume of the box

$$V = xyz \quad (18.34)$$

Subject to the two constraints

$$y + 2z = 20 \quad (18.35)$$

$$x + 2z = 6 \quad (18.36)$$

Substituting from the constraints into the Volume formula,

$$V = (6 - 2z)(20 - 2z)z = 4z^3 - 52z^2 + 120z \quad (18.37)$$

$$V' = 12z^2 - 104z + 120 \quad (18.38)$$

$$V'' = 24z - 104 \quad (18.39)$$

To maximize V we solve

$$0 = V' = 12z^2 - 104z + 120 = 4(3z^2 - 26z + 30) \quad (18.40)$$

From the quadratic equation we find that

$$z = \frac{1}{3} \left(13 \pm \sqrt{79} \right) \quad (18.41)$$

To determine which root, we evaluate the second derivative:

$$V'' \left(\frac{1}{3} \left(13 + \sqrt{79} \right) \right) = 24 \left(\frac{1}{3} \left(13 + \sqrt{79} \right) \right) - 104 = 8\sqrt{79} > 0 \quad (18.42)$$

$$V'' \left(\frac{1}{3} \left(13 - \sqrt{79} \right) \right) = 24 \left(\frac{1}{3} \left(13 - \sqrt{79} \right) \right) - 104 = -8\sqrt{79} < 0 \quad (18.43)$$

Since we want to maximize the volume, we required $V''(z) < 0$ to give a local maxi-

mum. So we choose the root with the minus sign:

$$z = \frac{1}{3} (13 - \sqrt{79}) \quad (18.44)$$

$$x = 6 - 2z = 6 - \frac{2}{3} (13 - \sqrt{79}) \quad (18.45)$$

$$y = 20 - 2z = 20 - \frac{2}{3} (13 - \sqrt{79}) \quad (18.46)$$

The volume is

$$\begin{aligned} V &= xyz \\ &= \left[6 - \frac{2}{3} (13 - \sqrt{79}) \right] \times \left[20 - 2z = 20 - \frac{2}{3} (13 - \sqrt{79}) \right] \\ &\quad \times \left[\frac{1}{3} (13 - \sqrt{79}) \right] \\ &= \frac{8}{27} (-442 + 79\sqrt{79}) \end{aligned} \quad (18.47)$$

Example 18.7 A car rental agency rents 210 cars per day at a rate of \$26/day. For each one-dollar increase in the daily rate, five fewer cars are rented. At what rate should the cars be rented to produce the maximum income, and what is the maximum income?

The total daily income (revenue) is

$$R = \text{income} = (\text{number of cars rented}) \times (\text{daily cost per car}) \quad (18.48)$$

Let the cost per car be p . Since the number of cars rented depends on the cost we write it as $N(p)$. Then the income is

$$R = pN(p) \quad (18.49)$$

We are also given the following data:

When $p = 26$, $N(p) = 210$

When $p = 27$, $N(p) = 210 - 5 = 205$

When $p = 28$, $N(p) = 210 - 5 - 5 = 200$

etc.

These points lie on a straight line. Taking the first two points:

$$(p_1, N_1) = (26, 210), \quad (p_2, N_2) = (27, 205)$$

we calculate the slope of the line:

$$m = \frac{N_2 - N_1}{p_2 - p_1} = \frac{205 - 210}{27 - 26} = -5 \quad (18.50)$$

Thus the equation of the line is

$$N(p) = N_1 + m(p - p_1) = 210 - 5(p - 26) = 340 - 5p \quad (18.51)$$

The daily income is then

$$R = pN(p) = p(340 - 5p) = 340p - 5p^2 \quad (18.52)$$

Differentiating,

$$R' = 340 - 10p \quad (18.53)$$

$$R'' = -10 \quad (18.54)$$

Since the second derivative is negative, the critical point is a local maximum. Solving

for the critical point,

$$0 = R' = 340 - 10p \implies p = \frac{340}{10} = 34 \quad (18.55)$$

So to maximize income, the company should charge \$34/day.

The number of cars sold at that rate will be

$$N(34) = 340 - 5(34) = 170 \quad (18.56)$$

The total income will be

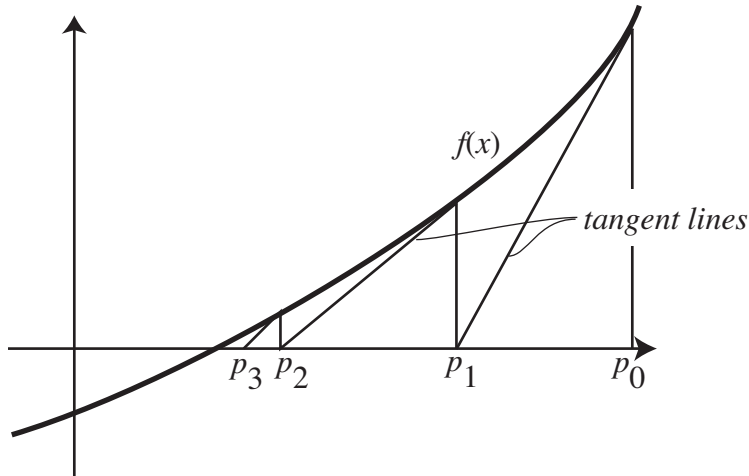
$$R = pN(p) = (34)(N(34)) = (34)(170) = \$5780 \quad (18.57)$$

Topic 19

Newton's Method

Newton's method uses the derivative to find the root of function.

To get the formula we need a first guess, say $x = p_0$. Then we project the tangent line at $(p_0, f(p_0))$ to the x -axis and use that value of the intersection of the tangent line with the x -axis as our next guess, as illustrated in the figure:



The slope of the tangent line $(p_0, f(p_0))$ is $f'(p_0)$.

The equation of the tangent line is the equation of a line with slope $m = f'(p_0)$

through the point $(p, f(p_0))$. Using the point-slope form:

$$y = y_1 + m(x - x_1) = f(p_0) + f'(p_0)(x - p_0) \quad (19.1)$$

gives the tangent line. This line intersects the x -axis when $y = 0$:

$$0 = f(p_0) + f'(p_0)(x - p_0) \implies x = p_0 - \frac{f(p_0)}{f'(p_0)} \quad (19.2)$$

We call this “improved” guess p_1 , and repeat the process, as follows:

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \quad (19.3)$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} \quad (19.4)$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)} \quad (19.5)$$

$$p_4 = \dots \quad (19.6)$$

and in general we have the Iteration Formula for Newton's Method:

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}} \quad (19.7)$$

Example 19.1 Find $\sqrt{2}$ using Newton's method to 3 decimal places starting with $p_0 = 2$.

We want to solve the equation $x^2 = 2$, which is equivalent to $x^2 - 2 = 0$. So we want to find the root of

$$y = x^2 - 2 \quad (19.8)$$

Then $y' = 2x$ so

$$p_{n+1} = p_n - \frac{p_n^2 - 2}{2p_n} = \frac{2p_n^2 - p_n^2 + 2}{2p_n} = \frac{p_n^2 + 2}{2p_n} = \frac{p_n}{2} + \frac{1}{p_n} \quad (19.9)$$

$$p_1 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} \approx 1.5 \quad (19.10)$$

$$p_2 = \frac{1.5}{2} + \frac{1}{1.5} \approx 1.41666 \quad (19.11)$$

$$p_3 = \frac{1.4166}{2} + \frac{1}{1.4166} \approx 1.4142 \quad (19.12)$$

$$p_4 = \frac{1.4142}{2} + \frac{1}{1.4142} \approx 1.4142 \quad (19.13)$$

Since the last two estimates are the same to 4 decimal places, the answer is 1.4142.

□.

Example 19.2 Find the value of x satisfies $x = \cos x$.

We need to find the root of $f(x) = x - \cos x$. Since $f'(x) = 1 + \sin x$, Newtons' formula gives

$$n(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x - \cos x}{1 + \sin x} = \frac{\cos x + x \sin x}{1 + \sin x} \quad (19.14)$$

Since we don't have any idea of where the root is any starting guess is as good as any other. Say we start with $x_0 = 1$. Then

$$x_1 = \frac{\cos 1 + (1) \sin 1}{1 + \sin 1} = \frac{0.540302 + .841471}{1 + .841471} = \frac{1.38177}{1.841471} = 0.750364 \quad (19.15)$$

$$x_2 = \frac{\cos .750364 + (.750364) \sin .750364}{1 + \sin .750364} = .739113 \quad (19.16)$$

$$x_3 = \frac{\cos .759113 + (.759113) \sin .739113}{1 + \sin .739113} = .739085 \quad (19.17)$$

$$x_4 = \frac{\cos .739085 + (.739085) \sin .739085}{1 + \sin .739085} = .739085 \quad (19.18)$$

So to three decimals $x = .739085$ radians.

Example 19.3 Find the root of

$$f(x) = x^3 - x \quad (19.19)$$

using Newton's method starting with $x = 2$, to 2 decimal places.

We observe that we could calculate the root exactly, by factoring:

$$0 = f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1) \quad (19.20)$$

So there are really three roots, at $x = -1, 0, 1$.

Let's see what Newton's method does. We are given $p_0 = 2$. Since

$$f'(x) = 3x^2 - 1 \quad (19.21)$$

Newton's Iteration Formula gives

$$p_{n+1} = p_n - \frac{x^3 - x}{3x^2 - 1} \quad (19.22)$$

Starting with $p_0 = 2$, we have

$$p_1 = 2 - \frac{2^3 - 2}{3(2^2) - 1} = 2 - \frac{6}{11} = \frac{16}{11} \approx 1.45455 \quad (19.23)$$

$$p_2 = 1.45 - \frac{(1.45)^3 - (1.45)}{3(1.45)^2 - 1} \approx 1.14 \quad (19.24)$$

$$p_3 = 1.14 - \frac{1.14^3 - (1.14)}{3(1.14)^2 - 1} \approx 1.02 \quad (19.25)$$

$$p_4 = 1.02 - \frac{1.02^3 - (1.02)}{3(1.02)^2 - 1} \approx 1.00 \quad (19.26)$$

$$p_5 = 1.00 - \frac{1.00^3 - (1.00)}{3(1.00)^2 - 1} = 1 \quad (19.27)$$

If we wanted to find the other roots we'd have to start with different values of p_0 .

Example 19.4 Use Newton's method to estimate the root of $x^4 = 1 + x$ to three decimal places.

We first write the function as

$$g(x) = x^4 - 1 - \frac{x}{2} = 0 \quad (19.28)$$

To estimate where the roots are we find extrema:

$$g'(x) = 4x^3 - \frac{1}{2} \quad (19.29)$$

So that $g'(x) = 0 \implies 4x^3 = 1/2 \implies x^3 = 1/8 \implies x = 1/2$.

Then we observe that as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. Since there is only one extrema, we expect it to be a minimum. This is confirmed by $g''(x) = 12x^2 > 0$.

Next we find the value of the minimum. If it is positive, there are no roots, and if it is negative there must be two roots. (Why?)

$$g(1/2) = (1/2)^4 - 1 - (1/2)/2 = \frac{1}{16} - 1 - \frac{1}{4} = \frac{1 - 16 - 4}{16} = \frac{-19}{16} < 0. \quad (19.30)$$

By the intermediate value theorem there are two roots, one each in the interval $(-\infty, 1/2)$ and $(1/2, \infty)$.

Newton's iteration formula is then

$$n(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - 1 - x/2}{4x^3 - 1/2} = x - \frac{2x^4 - x - 2}{8x^3 - 1} = \frac{2(3x^4 + 1)}{8x^3 - 1} \quad (19.31)$$

If we start with $p_0 = 1$, then

$$p_1 = \frac{2(3(1) + 1)}{8(1) - 1} = \frac{8}{7} \approx 1.14286 \quad (19.32)$$

$$p_2 = \frac{2(3(1.14286^4) + 1)}{8(1.14286^3) - 1} \approx 1.11827 \quad (19.33)$$

$$p_3 = \frac{2(3(1.11827^4) + 1)}{8(1.11827^3) - 1} \approx 1.11735 \quad (19.34)$$

$$p_4 = \frac{2(3(1.11735^4) + 1)}{8(1.11735^3) - 1} \approx 1.11735 \quad (19.35)$$

Since the last two estimates agree to 5 decimal places, one root is 1.11735 to five decimal places. or 1.117 to three decimal places.

To find the other root we need to use a different starting guess. If we try $p_0 = -1$ we get

$$p_1 = -0.888889 \quad (19.36)$$

$$p_2 = -0.868117 \quad (19.37)$$

$$p_3 = -0.867471 \quad (19.38)$$

$$p_4 = -0.867471 \quad (19.39)$$

so to three decimal places the second root is at -0.867.

Topic 20

Antiderivatives

Definition 20.1 A function F is called an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all x in I .

Example 20.1 $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

The reason why we say “an” antiderivative rather than “the” antiderivative is because we can add any arbitrary constant to the antiderivative:

Example 20.2 $F(x) = x^3 + 17$ is also an antiderivative of $f(x) = 3x^2$.

Theorem 20.2 If F is any antiderivative of $f(x)$ on an interval I then the most general antiderivative of f on I can be written as

$$F(x) + C \tag{20.1}$$

where C is an arbitrary constant.

We use the **integral** symbol \int to represent the antiderivative, as follows: If $F(x)$ is an antiderivative of $f(x)$ then we write

$$\int f(x)dx = F(x) + C \quad (20.2)$$

and

$$F'(x) = f(x) \implies F(x) = \int F'(x)dx + C' \quad (20.3)$$

where C and C' are arbitrary constants.

Example 20.3 Find the most general antiderivative of $f(x) = \sin x$

Let $F(x) = -\cos x$. Then $F'(x) = f(x)$. Hence the general form is

$$F(x) = -\cos x + C \quad \square \quad (20.4)$$

In terms of the integral notation

$$\int \sin x dx = -\cos x + C \quad (20.5)$$

We have the following properties for antiderivatives.

$$\int C f(x)dx = C \int f(x)dx \quad (20.6)$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx \quad (20.7)$$

Example 20.4 Find the most general antiderivative of $f(x) = x - 3$.

We need to find a function whose derivative is $x - 3$. Since $F(x) = x^2/2 - 3x$ has such a derivative, the most general antiderivative is

$$F(x) = \frac{x^2}{2} - 3x + C \quad \square \quad (20.8)$$

or

$$\int (x - 3)dx = \frac{x^2}{2} - 3x + C \quad (20.9)$$

Example 20.5 Find the most general form of the antiderivative of $f(x) = x^n, n \geq 0$.

Let $F(x) = \frac{x^{n+1}}{n+1}$. Then since $f'(x) = x^n$, we see that $F(x)$ is an antiderivative of x^n . The most general antiderivative is

$$F(x) = \frac{x^{n+1}}{n+1} + C \quad (20.10)$$

i.e.,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (20.11)$$

Example 20.6 Find the most general antiderivative of $f(x) = 10/x^9$.

We need to find a function whose derivative is $f(x)$, i.e., we need to find some function $F(x)$ such that

$$F'(x) = \frac{10}{x^9} = 10x^{-9} \quad (20.12)$$

Then

$$F(x) = \int 10x^{-9} dx = \frac{10x^{-9+1}}{-9+1} + C = -\frac{10x^{-8}}{8} + C = -\frac{5x^{-8}}{4} + C \quad (20.13)$$

Example 20.7 Find $f(x)$ if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

$f'(x) = x\sqrt{x}$ is equivalent to

$$f(x) = \int f'(x)dx + C \quad (20.14)$$

$$= \int x\sqrt{x}dx + C \quad (20.15)$$

$$= \int x^{3/2}dx + C \quad (20.16)$$

$$= \frac{x^{5/2}}{5/2} + C = \frac{2x^{5/2}}{5} + C \quad (20.17)$$

To find the value of C substitute the condition $f(1) = 2$.

$$2 = f(1) = \frac{2}{5} + C \implies C = 2 - \frac{2}{5} = \frac{8}{5} \quad (20.18)$$

Hence

$$f(x) = \frac{2x^{5/2}}{5} + \frac{8}{5} \quad \square \quad (20.19)$$

Example 20.8 Solve $f'(x) = 1 - 6x$, $f(0) = 8$.

$$f(x) = \int f'(x)dx + C = \int (1 - 6x)dx + C \quad (20.20)$$

$$= \int dx - 6 \int xdx + C \quad (20.21)$$

$$= x - 3x^2 + C \quad (20.22)$$

Substituting $f(0) = 8$ gives

$$8 = C \quad (20.23)$$

hence

$$f(x) = x - 3x^2 + 8 \quad \square \quad (20.24)$$

Each derivative we know gives us a formula for an antiderivative, as summarized here.

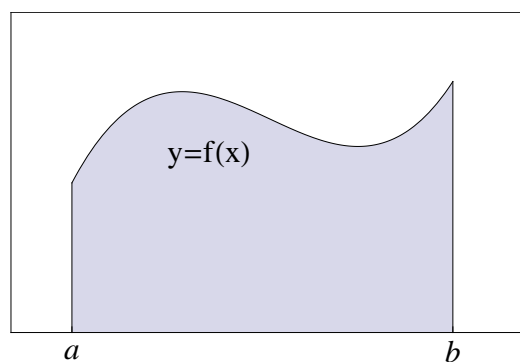
Function	Antiderivative
x^n ($n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Topic 21

Areas and Distances

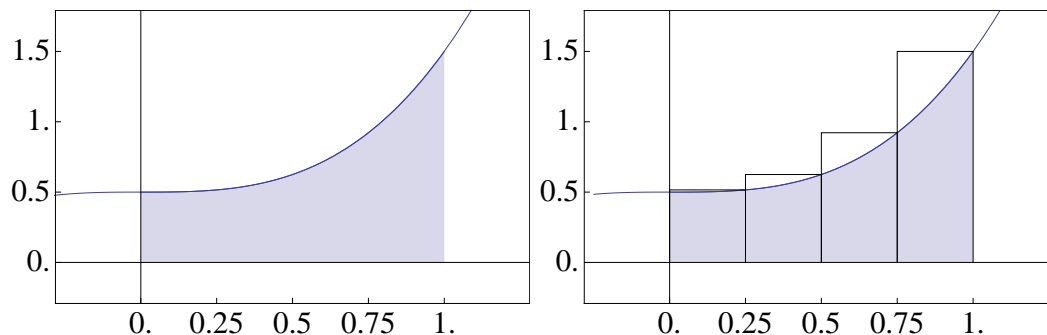
Areas

How do we find the area between a curve $y = f(x)$ and the x -axis in the region between $x = a$ and $x = b$?



We begin by approximating the area by trying to fill up the space with rectangles, and then adding up the area of each small rectangle.

Suppose we want to find the area under the curve $y = x^3 + 1/2$ on $[0, 1]$, as illustrated below on the left. We fill in the area with rectangles whose upper-left-hand corner just touch the function, as shown in the right. A small part of the rectangle sticks out but we are only trying to get an estimate, so our estimate will be a bit large.



The area of each rectangle is its base \times height. If we use 4 intervals, as shown above to the right, the area of the rectangles is then (remember that $f(x) = 1/2 + x^3$ in this example),

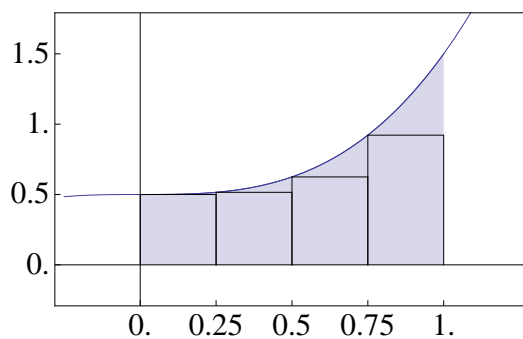
$$A = 0.25 \times f(0.25) + 0.25 \times f(0.5) + 0.25 \times f(0.75) + 0.25 \times f(1) \quad (21.1)$$

$$\begin{aligned} &= (0.25)(1/2 + (0.25)^3) + (0.25)(1/2 + (0.5)^3) + (0.25)(1/2 + (0.75)^3) \\ &\quad + (0.25)(1/2 + (1)^3) \end{aligned} \quad (21.2)$$

$$= (0.25)(0.51625) + (0.25)(0.625) + (0.25)(0.921875) + 1.5 \quad (21.3)$$

$$= 0.890625 \quad (21.4)$$

Now let us repeat the process using the upper left hand corner of rectangles, rather than the upper right hand corner:



In this case the rectangles give an underestimate:

$$A = 0.25f(0) + 0.25f(.5) + 0.25f(.5) + 0.25f(.75) \tag{21.5}$$

$$= 0.25(0.5) + (0.25)(0.51625) + (0.25)(0.625) + (0.25)(0.921875) \tag{21.6}$$

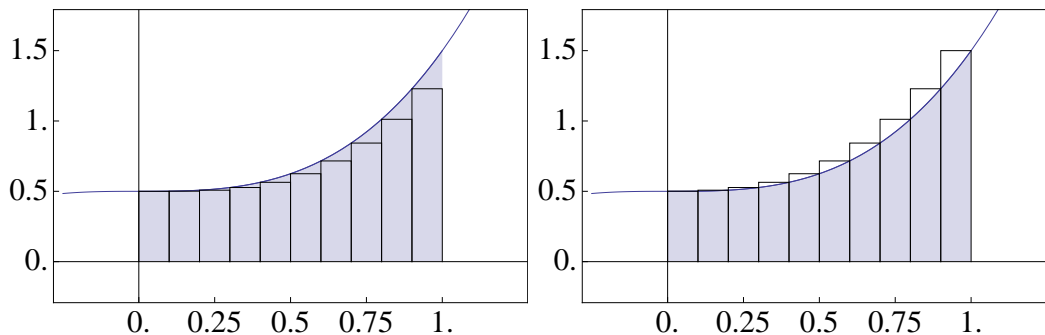
$$= 0.640625 \tag{21.7}$$

Based on these two estimates we know the area is somewhere in the range

$$0.640625 < A < 0.890625 \tag{21.8}$$

If we want to get better estimates we use smaller and smaller intervals. Suppose that

instead of 4 intervals we use 10:



Then we get:

$$A_{\text{Left}} = 0.10[f(0) + f(0.1) + \cdots + f(0.9)] \quad (21.9)$$

$$\begin{aligned} &= 0.10(0.5 + 0.501 + 0.508 + 0.527 + 0.564 + 0.625 + 0.716 + 0.843 \\ &\quad + 1.012 + 1.229) = 0.702 \end{aligned} \quad (21.10)$$

$$A_{\text{Right}} = 0.10[f(0.1) + f(0.2) + \cdots + f(1.0)] \quad (21.11)$$

$$\begin{aligned} &= 0.10(0.501 + 0.508 + 0.527 + 0.564 + 0.625 + 0.716 + 0.843 \\ &\quad + 1.012 + 1.229 + 1.5) = 0.803 \end{aligned} \quad (21.12)$$

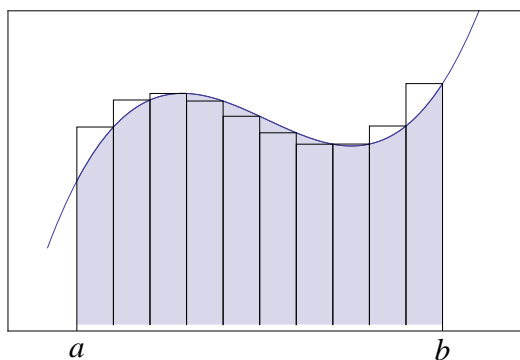
This says that

$$.702 < A < .803 \quad (21.13)$$

which gives a more accurate approximation.

As you might guess, the approximation gets tighter the more intervals you get, and the right and the left approximations get closer and closer to each other. In the limit as the interval width goes to zero (and the number of intervals goes to ∞) then they approach the same answer.

We cannot, in general, make the assumption that the left rectangles will underestimate the area and right rectangles will overestimate it, as in the following figure, we see that some rectangles are too large and some are too small:



The general principle, though, still holds: as the interval width $\rightarrow 0$ and $n \rightarrow \infty$, both the left and right approximations approach the same limit.

Let us call the width of each rectangle Δx .

$$\Delta x = \frac{b - a}{n} \quad (21.14)$$

Then if the corners are at x_0, x_1, \dots, x_n , the left rectangles use x_0, \dots, x_{n-1} and the right rectangles use x_1, \dots, x_n :

$$A_{\text{Left}} \approx \Delta x [f(x_0) + f(x_1) + \dots + f(x_{n-1})] = \sum_{i=0}^{n-1} \Delta x f(x_i) \quad (21.15)$$

$$A_{\text{Right}} \approx \Delta x [f(x_1) + f(x_2) + \dots + f(x_n)] = \sum_{i=1}^n \Delta x f(x_i) \quad (21.16)$$

In each sum there are n terms.

Suppose instead of the endpoints we choose **any point in each interval**:

$$x_1 \text{ in } [a, \Delta x]$$

$$x_2 \text{ in } [a + \Delta x, a + 2\Delta x]$$

$$\vdots$$

$$x_n \text{ in } [a + (n - 1)\Delta x, a + n\Delta x]$$

The left-endpoints correspond to choosing

$$x_i = a + (i - 1) * \Delta x \tag{21.17}$$

and the right endpoints correspond to choosing

$$x_i = a + i * \Delta x \tag{21.18}$$

In either case we still have

$$(i - 1) * \Delta x \leq x_i \leq i * \Delta x \tag{21.19}$$

Then the area approximation can be written as

$$A \approx \sum_{i=1}^n \Delta x f(x_i) = \sum_{i=1}^n \frac{b-a}{n} f(x_i) \tag{21.20}$$

It can be proven that the following limit converges, regardless of where we choose the points:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x f(x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} f(x_i) \quad (21.21)$$

Distances Traveled

If we travel at some velocity for a certain amount of time then

$$\text{distance traveled} = (\text{velocity}) \times (\text{time}) \quad (21.22)$$

If the velocity is changing, it is more difficult to calculate the distance traveled.

Suppose that the velocity is given by some function

$$v = f(t) \quad (21.23)$$

Then during the time interval $t = 0$ to $t = \Delta t$ we travel a distance

$$d_1 \approx f(0)\Delta t \quad (21.24)$$

during the time from $t = \Delta t$ to $t = 2\Delta t$ we travel a distance

$$d_2 \approx f(\Delta t)\Delta t \quad (21.25)$$

during the time from $t = 2\Delta t$ to $t = 3\Delta t$ we travel a distance

$$d_3 \approx f(2\Delta t)\Delta t \quad (21.26)$$

and so forth. The total distance traveled from $t = 0$ to $t = n\Delta t$

$$d = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^n f(t_i)\Delta t \quad (21.27)$$

where each

$$(n-1) * \Delta t \leq t_i \leq n * \Delta t \quad (21.28)$$

This is the same as the area under the curve of $f(t)$ on the interval $[0, n\Delta t]$.

Example 21.1 Estimate the area under the curve $y = 1/x^2$ from $x = 1$ to $x = 5$ using left and right endpoints and 4 rectangles.

Since $\Delta x = \frac{5-1}{4} = 1$, using left endpoints we find that

$$A = (1) \times [f(1) + f(2) + f(3) + f(4)] \quad (21.29)$$

$$= 1 + .25 + .11111 + .0625 = 1.42361 \quad (21.30)$$

Using right endpoints,

$$A = (1) \times [f(2) + f(3) + f(4) + f(5)] \quad (21.31)$$

$$= 0.25 + 0.11111 + 0.0625 + 0.04 = 0.463611 \quad (21.32)$$

Topic 22

Definite Integrals

We define the **definite integral** as the limit of the sum that gives the area that we found in the last section:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x f(x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} f(x_i) = \int_a^b f(x) dx \quad (22.1)$$

We call the numbers on the integral sign the **upper limit** (b) and **lower limit** (a).

Thus

$$\int_a^b f(x) dx = \text{Area between the curve of } f(x) \text{ and the x-axis on } [a, b] \quad (22.2)$$

If the function $f(x)$ is negative, then the area is defined to be negative.

The general term for the sum is **Riemann Sum**. A Riemann Sum estimates an

integral by taking any arbitrary point inside each interval. Both left- and right-rectangle approximations are examples of Riemann Sum.

One common formula for estimating the definite integral called the **Midpoint Rule**. Instead of using left or right endpoints, the midpoint rule uses the center of each interval.

$$x_1 = 0.5\Delta x \quad (22.3)$$

$$x_2 = 1.5\Delta x \quad (22.4)$$

$$x_3 = 2.5\Delta x \quad (22.5)$$

$$:x_i = \left(i - \frac{1}{2}\right) \times \frac{b - a}{n}$$

The Midpoint Rule is another example of a Riemann Sum.

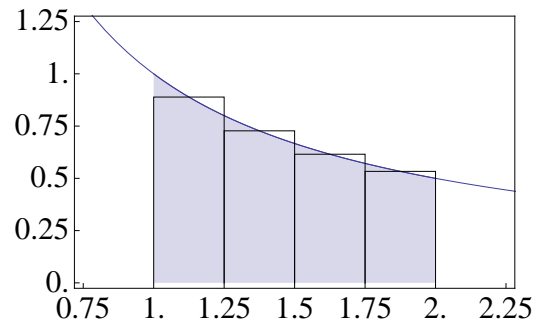
$$\int_a^b f(x)dx = \sum_{i=1}^n f(x_i)\Delta x = \frac{b - a}{n} \sum_{i=1}^n f\left(\left(i - .5\right) * \Delta x\right) \quad (22.6)$$

The midpoint rule is generally more accurate than either the left or right rectangles because it takes the average of the two endpoints and errors tend to cancel one-another out.

Example 22.1 Use the midpoint rule to estimate

$$\int_1^2 \frac{1}{x} dx \quad (22.7)$$

using 4 intervals.



Since there are $n = 4$ intervals, the width of each interval is

$$\Delta x = \frac{b - a}{n} = \frac{2 - 1}{4} = 0.25 \quad (22.8)$$

Therefore the left endpoints are at

$$1, 1.25, 1.5, 1.75 \quad (22.9)$$

as illustrated on the graph. Therefore the midpoints are at

$$1.125, 1.375, 1.625, 1.875 \quad (22.10)$$

We estimate the area as

$$\int_1^2 \frac{1}{x} dx \approx (0.25)[f(1.125) + f(1.375) + f(1.625) + f(1.875)] \quad (22.11)$$

$$\approx 0.25(0.88888 + 0.727273 + 0.61538 + 0.533333) \approx 0.69122 \quad (22.12)$$

Properties of the Definite Integral

$$1. \int_a^b f(x)dx = - \int_b^a f(x)dx$$

This is because on the left $\Delta x = \frac{b-a}{n}$ and on the right $\Delta x = \frac{a-b}{n}$.

$$2. \int_a^a f(x)dx = 0$$

This is because the area under a point is zero.

$$3. \int_a^b Cdx = C(b-a), \text{ where } C \text{ is any constant}$$

$$4. \int_a^b Cf(x)dx = C \int_a^b f(x)dx, C \text{ is any constant.}$$

$$5. \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$6. \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 22.2 Find $\int_0^5 (1+3x)dx$

$$\int_0^5 (1+3x)dx = \int_0^5 1 \times dx + \int_0^5 3xdx \quad \text{property 5} \quad (22.13)$$

$$= 1(5-0) + 3 \int_0^5 xdx \quad \text{properties 3 and 4} \quad (22.14)$$

$$= 5 + 3 \int_0^5 xdx \quad (22.15)$$

$$= 5 + 3 * \frac{1}{2}(5 \times 5) = 42.5 \quad (22.16)$$

where the last line follows because $\int_0^5 xdx$ is the area of a triangle of base 5 and height

5. Hence

Topic 23

Fundamental Theorem of Calculus

The fundamental theorem of calculus relates the area under a curve to derivatives and anti-derivatives. Let $f(x)$ be defined on $[a, b]$ and define the function

$$A(x) = \int_a^x f(t)dt \quad (23.1)$$

where $a \leq x \leq b$. We are interested in calculating

$$\frac{dA}{dx} = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} \quad (23.2)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right] \quad (23.3)$$

From property 6 of definite integrals

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt \quad (23.4)$$

Setting $b = x + h$ and $c = x$ gives

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt \quad (23.5)$$

hence

$$\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt \quad (23.6)$$

and therefore

$$\frac{dA}{dx} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right] \quad (23.7)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t)dt \quad (23.8)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \times (f(x) \times h) \quad (\text{Rectangle approximation}) \quad (23.9)$$

$$= \lim_{h \rightarrow 0} f(x) = f(x) \quad (23.10)$$

In other words, A is an antiderivative of $f(x)$.

Theorem 23.1 Fundamental Theorem Calculus: First Form. Let $f(x)$ be continuous on $[a, b]$ and define

$$A(x) = \int_a^x f(t)dt \tag{23.11}$$

where $a \leq t \leq b$. Then $A(x)$ is continuous on $[a, b]$ and

$$A'(x) = f(x) \tag{23.12}$$

If we let $F(x) = A(x) + C$ for any constant C , then $F'(x) = A'(x)$ so that $F'(x) = f(x)$ in the fundamental theorem. If we substitute into the formula for the fundamental theorem of calculus, we get

$$F(x) - C = \int_a^x f(t)dt \tag{23.13}$$

Hence

$$F(a) - C = \int_a^a f(t)dt = 0 \tag{23.14}$$

so that

$$C = F(a) \tag{23.15}$$

hence

$$F(x) = C + \int_a^x f(t)dt = F(a) + \int_a^x f(t)dt \tag{23.16}$$

This gives us the second form of the fundamental theorem of calculus

Theorem 23.2 Fundamental Theorem of Calculus: Second Form. Let $f(x)$ be continuous on $[a, b]$ and let $F(x)$ be any antiderivative of f . Then

$$\int_a^b f(x)dx = F(b) - F(a) \quad (23.17)$$

This form relates the antiderivative to the definite integral.

Example 23.1 Find $\int_{-2}^3 3x^3 dx$.

$$\int_{-2}^3 x^3 dx = \frac{x^4}{4} \Big|_{-2}^3 = \frac{(3)^4}{4} - \frac{(-2)^4}{4} = \frac{81}{4} - \frac{16}{4} = \frac{65}{4} \quad (23.18)$$

Example 23.2 $\int_0^1 x^{4/5} dx$

Example 23.3 $\int_0^4 (1 + 3y - y^2) dy$

Example 23.4 Find the area under the parabola $y = x^2$ from $x = 3$ to $x = 4$.

$$\int_3^4 x^2 dx = \frac{x^3}{3} \Big|_3^4 = \frac{4^3 - 3^3}{3} = \frac{64 - 27}{3} = \frac{37}{3} \quad (23.19)$$

Example 23.5 Find the area under the curve of $y = \sqrt[3]{x}$ from $x = 0$ to $x = 27$ (Ans: 243/4)

Example 23.6 Find the area under the curve $y = \sin x$, from $x = 0$ to $x = \pi$.

Using the relationship $f(x) = F'(x)$ in the fundamental theorem of calculus gives a third form.

Theorem 23.3 Net Change Theorem: Fund. Thm. of Calculus, 3rd Form.

$$\int_a^b F'(t)dt = F(b) - F(a) \quad (23.20)$$

Taking the upper limit as x ,

$$\int_a^x F'(t)dt = F(x) - F(a) \quad (23.21)$$

Differentiating

$$\frac{d}{dx} \int_a^x F'(t)dt = \frac{d}{dx}(F(x) - F(a)) = F'(x) \quad (23.22)$$

or

$$\boxed{\frac{d}{dx} \int_a^x f(t)dt = f(x)} \quad (23.23)$$

Example 23.7 Water flows from the bottom of a storage tank at a rate of

$$r(t) = 200 - 4t \text{ liters/minute}, \quad 0 \leq t \leq 50 \quad (23.24)$$

Find the amount of water that flows from the rod during the first 10 minutes.

Let $V(t)$ be the volume at any time.

The rate at which water flows is the change in volume:

$$\frac{dV}{dt} = r(t) = 200 - 4t \quad (23.25)$$

Then $V(0)$ is the amount of water in the tank at $t = 0$.

$V(10)$ is the amount of water left in the tank at $t = 10$.

The amount of water that flows out during the 10 minutes is $V(0) - V(10)$.

By the net change theorem,

$$V(10) - V(0) = \int_0^{10} dV dt = \int_0^{10} (200 - 4t) dt \quad (23.26)$$

$$= 200t - 2t^2 \Big|_0^{10} \quad (23.27)$$

$$= 200(10) - 2(100)^2 \quad (23.28)$$

$$= 2000 - 20,000 = -18,000 \quad (23.29)$$

Thus 18,000 liters of water flow from the tank.

Since the fundamental theorem of calculus says that

$$\int_a^b f(x)dx = F(b) - F(a) \quad (23.30)$$

where F is any antiderivative of f , we can write

$$\int_a^x f(t)dt = F(x) - F(a) \quad (23.31)$$

Differentiating with respect to x ,

$$\frac{d}{dx} \int_a^x f(t)dt = \frac{d}{dx} [F(x) - F(a)] = F'(x) - 0 = F'(x) = f(x) \quad (23.32)$$

where the last step follows because F is an antiderivative of f . Hence we have

Theorem 23.4 Fundamental Theorem of Calculus, 4th form

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \quad (23.33)$$

Example 23.8

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \frac{1}{x^2 + 1} \quad (23.34)$$

Example 23.9

$$\frac{d}{dx} \int_x^1 \cos \sqrt{t} dt = -\frac{d}{dx} \int_1^x \cos \sqrt{t} dt = \cos \sqrt{x} \quad (23.35)$$

We can also derive a chain rule for differentiating integrals:

$$\frac{d}{dx} \int_0^{g(x)} f(t) dt = \frac{d}{dx} \int_0^u f(t) dt \quad (23.36)$$

where $u = g(x)$. If we define the function

$$I(u) = \int_0^u f(t) dt \quad (23.37)$$

Then

$$\frac{d}{dx} \int_0^{g(x)} f(t) dt = \frac{d}{dx} I(g(x)) = I'(g(x))g'(x) \quad (23.38)$$

Thus we have

Theorem 23.5 Fundamental Theorem of Calculus, Chain Rule Form

$$\frac{d}{dx} \int_0^{g(x)} f(t) dt = \left(\frac{d}{du} \int_0^u f(t) dt \right) \Big|_{u=g(x)} \times \left(\frac{d}{dx} g(x) \right) \quad (23.39)$$

Example 23.10 $\frac{d}{dx} \int_0^{1/x} \sin^4 t dt$

We define $u = g(x) = 1/x$. Then $g'(x) = -1/x^2$, so that

$$\begin{aligned} \frac{d}{dx} \int_0^{1/x} \sin^4 t dt &= -\frac{1}{x^2} \left(\frac{d}{du} \int_0^u \sin^4 t dt \right) \Big|_{u=g(x)=1/x} \\ &= -\frac{1}{x^2} \sin^4 \frac{1}{x} \end{aligned}$$

Example 23.11

$$\frac{d}{dx} \int_x^{x^2} \sqrt{1+t^2} dt = \frac{d}{dx} \int_x^c \sqrt{1+t^2} dt + \frac{d}{dx} \int_c^{x^2} \sqrt{1+t^2} dt \quad (23.40)$$

for any constant c . Hence

$$\begin{aligned} \frac{d}{dx} \int_x^{x^2} \sqrt{1+t^2} dt &= -\frac{d}{dx} \int_c^x \sqrt{1+t^2} dt + \frac{d}{dx} \int_c^{x^2} \sqrt{1+t^2} dt \\ &= -\sqrt{1+x^2} + \left(\frac{d}{du} \int_c^u \sqrt{1+t^2} dt \right) \Big|_{u=x^2} \times \left(\frac{d}{dx} x^2 \right) \\ &= -\sqrt{1+x^2} + \left(\sqrt{1+u^2} \right) \Big|_{u=x^2} \times (2x) \\ &= -\sqrt{1+x^2} + 2x\sqrt{1+x^4} \end{aligned}$$

Moving Particles

Let $s(t)$ describe the position of a particle as a function of time and $v(t)$ be the velocity. Then

$$v(t) = s'(t) \tag{23.41}$$

By the net change theorem we have the **displacement of the particle**

$$\boxed{\text{Displacement} = \int_{t_1}^{t_2} v(t) dt = \int_{t_1}^{t_2} s'(t) dt = s(t_2) - s(t_1)} \tag{23.42}$$

The **total distance traveled** is

$$\boxed{\text{Distance Traveled} = \int_{t_1}^{t_2} |v(t)| dt} \tag{23.43}$$

Example 23.12 Let the velocity function be $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled on $0 \leq t \leq 6$.

$$\begin{aligned} \text{displacement} &= s(6) - s(0) \\ &= \int_0^6 v(t) dt = \int_0^6 (t^2 - 2t - 8) dt \\ &= \left. \frac{t^3}{3} - t^2 - 8t \right|_0^6 \\ &= \frac{216}{3} - 36 - 8(6) = -12 \end{aligned}$$

To get the distance traveled we need the absolute value of $v(t)$. But

$$v(t) = t^2 - 2t - 8 = (t - 4)(t + 2) \quad (23.44)$$

So $v(t) < 0$ when either of the following are true:

(a) $t - 4 < 0$ and $t + 2 > 0$. This means $t < 4$ and $t > -2$ or $-2 < t < 4$.

(b) $t - 4 > 0$ and $t + 2 < 0$. This means $t > 4$ and $t < -2$ at the same time, which never happens.

For the interval $[0, 6]$ we have then that $v(t) < 0$ on $[0, 4]$ and $v(t) > 0$ on $[4, 6]$.

Using the definition of absolute value:

$$|v(t)| = \begin{cases} v(t), & v(t) \geq 0 \\ -v(t), & v(t) < 0 \end{cases} \quad (23.45)$$

Therefore the distance traveled is

$$\text{distance traveled} = \int_0^6 |v(t)| dt = \int_0^4 (-v(t)) dt + \int_4^6 v(t) dt \quad (23.46)$$

$$= - \int_0^4 (t^2 - 2t - 8) dt + \int_4^6 (t^2 - 2t - 8) \quad (23.47)$$

$$= - \left(\frac{t^3}{3} - t^2 - 8t \Big|_0^4 \right) + \left(\frac{t^3}{3} - t^2 - 8t \Big|_4^6 \right) \quad (23.48)$$

$$= - \left(\frac{4^3}{3} - 4^2 - 8(4) \right) + \left[\left(\frac{6^3}{3} - 6^2 - 8(6) \right) - \left(\frac{4^3}{3} - 4^2 - 8(4) \right) \right] \quad (23.49)$$

$$= \frac{80}{3} - 12 + \frac{80}{3} = \frac{120}{3} \quad (23.50)$$

Topic 24

Substitution

The method of substitution allows us to find new integral formulas. The idea is to change the name of variable and make a substitution into the original formula. We outline the method by example. Suppose we want to solve

$$\int 2x\sqrt{1+x^2}dx \tag{24.1}$$

If we define a new variable

$$u = 1 + x^2 \tag{24.2}$$

Then

$$du = 2xdx \implies dx = \frac{du}{2x} \tag{24.3}$$

Hence

$$\int 2x\sqrt{1+x^2}dx = \int 2x\sqrt{u}\frac{du}{2x} = \int \sqrt{u}du \quad (24.4)$$

$$= \int u^{1/2}du \quad (24.5)$$

$$= \frac{u^{3/2}}{3/2} + C \quad (24.6)$$

$$= \frac{2}{3}(1+x^2)^{3/2} + C \quad (24.7)$$

The general method is to look for a way to write the integral as

$$\int F'(g(x))g'(x)dx = F(g(x)) + C \quad (24.8)$$

This works because by the chain rule

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) \quad (24.9)$$

so that

$$\int F'(g(x))g'(x)dx = \int \frac{d}{dx}[F(g(x))]dx = F(g(x)) + C \quad (24.10)$$

Example 24.1 Find

$$\int x^5 \cos(x^6 + 7) dx$$

We make the substitution

$$u = x^6 + 7 \tag{24.11}$$

$$\implies du = 6x^5 dx \tag{24.12}$$

$$\implies x^5 dx = du/6 \tag{24.13}$$

$$\int x^5 \cos(x^6 + 7) dx = \int \cos(x^6 + 7) \times (x^5 dx) \tag{24.14}$$

$$= \int (\cos u) \times (du/6) \tag{24.15}$$

$$= \frac{1}{6} \int \cos u du = \frac{1}{6} \sin u + C \tag{24.16}$$

$$= \frac{1}{6} \sin(x^6 + 7) + C \tag{24.17}$$

Example 24.2 Solve

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

We let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin u du = -\cos u + C = -\cos \sqrt{x} + C \tag{24.18}$$

Example 24.3 Find

$$\int \sqrt{4t-1} dt$$

Here we have two choices, we can either use $u = 4t - 1$ or $u = \sqrt{4t - 1}$.

Method 1: Let $u = 4t - 1$. Then $du = 4dt$.

$$\int \sqrt{4t-1} dt = \int \sqrt{u} \times \frac{dt}{4} = \frac{1}{4} \int u^{1/2} du = \frac{1}{4(3/2)} u^{3/2} + C = \frac{1}{6} (4t-1)^{3/2} + C \quad (24.19)$$

Method 2: Let $u = \sqrt{4t - 1}$. Then by the chain rule

$$du = \frac{1}{2\sqrt{4t-1}} \times 4dt = \frac{2dt}{\sqrt{4t-1}} = \frac{2dt}{u} \implies dt = \frac{udu}{2} \quad (24.20)$$

$$\int \sqrt{4t-1} dt = \int u \times \frac{udu}{2} = \frac{1}{2} u^2 du = \frac{1}{6} u^3 + C = \frac{1}{6} (4t-1)^{3/2} + C \quad (24.21)$$

The procedure is similar for definite integrals, only we must remember that the limits apply to the old variable: we must change the limits to apply to the new variable.

Example 24.4 Find $\int_0^2 (2x-1)^{25} dx$.

let $u = 2x - 1$. Then $du = 2dx$.

When $x = 0$, $u = 2 \times 0 - 1 = -1$

When $x = 1$, $u = 2 \times 1 - 1 = 1$

$$\int_0^1 (2x-1)^{25} dx = \int_{-1}^1 u^{25} du = \frac{u^{26}}{26} \Big|_{-1}^1 = \frac{1}{26} 1 - 1 = 0 \quad (24.22)$$

Example 24.5 $\int_0^{\sqrt{\pi/2}} x \cos(x^2) dx$

Let $u = x^2$; then $du = 2x dx$ or $x dx = du/2$.

When $x = 0$, $u = 0^2 = 0$.

When $x = \sqrt{\pi/2}$, $u = (\sqrt{\pi/2})^2 = \pi/2$

$$\int_0^{\sqrt{\pi/2}} x \cos(x^2) dx = \frac{1}{2} \int_0^{\pi/2} \cos u du \quad (24.23)$$

$$= \frac{1}{2} \sin u \Big|_0^{\pi/2} \quad (24.24)$$

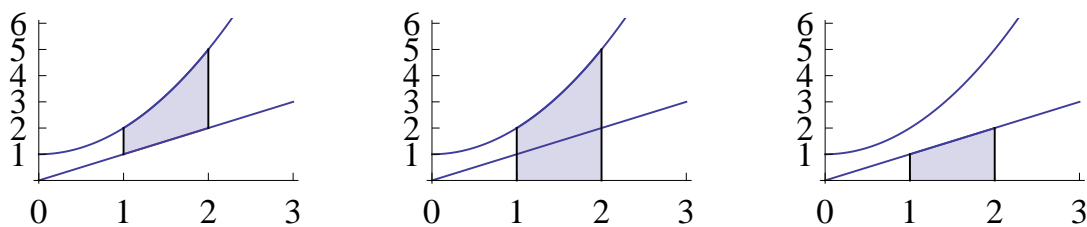
$$= \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin 0 \quad (24.25)$$

$$= \frac{1}{2} \quad (24.26)$$

Topic 25

Areas between curves

We can get the area between two positive curves by observing that the area is the difference of the area under the top curve and the area under the lower curve.



As we see in the illustration,

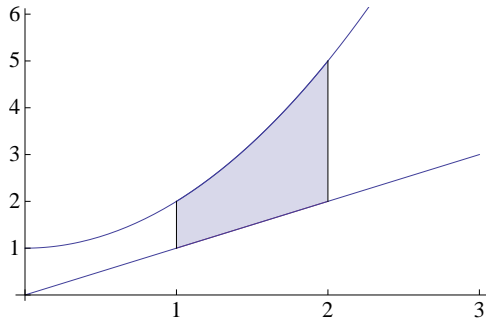
$$\text{Area Between Curves} = \text{Area under Top Curve} - \text{Area under Bottom Curve}$$

Hence

$$\text{Area} = \int_a^b (f(x) - g(x)) dx \quad (25.1)$$

So long as $f(x) \geq g(x) \geq 0$ for all x in (a, b)

Example 25.1 Find the area between $y = x^2 + 1$, $y = x$, $x = 1$, and $x = 2$. This area is illustrated in the figure:



The upper curve is $f(x) = x^2 + 1$.

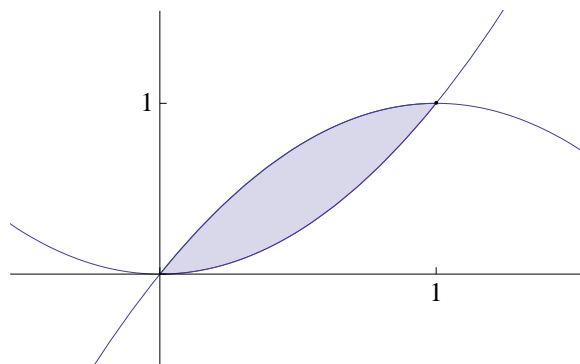
The lower curve is $g(x) = x$.

The left and right edges are the straight vertical lines $x = 1$ and $x = 2$

$$\begin{aligned}
 A &= \int_1^2 ((x^2 + 1) - x) dx \\
 &= \left. \frac{1}{3}x^3 + x - \frac{1}{2}x^2 \right|_1^2 \\
 &= \left(\frac{1}{3}(8) + 2 - \frac{1}{2}(4) \right) - \left(\frac{1}{3}(1) + 1 - \frac{1}{2}(1) \right) \\
 &= \frac{8}{3} + 2 - 2 - \frac{1}{3} - 1 + \frac{1}{2} = \frac{11}{6}
 \end{aligned}$$

Example 25.2 Find the area between the curves $y = 2x - x^2$ and $y = x^2$.

These two curves form a closed area between them, as illustrated in the figure:



The top curve is $y = 2x - x^2$.

The bottom curve is $y = x^2$.

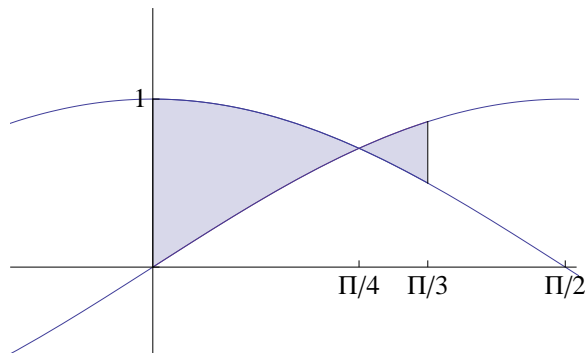
They intersect when

$$2x - x^2 = x^2 \implies 2x - 2x^2 = 0 \implies 2x(1 - x) = 0 \implies x = 0, x = 1 \quad (25.2)$$

Substituting for $y = x^2$ at the intersection points, they are $(0,0)$ and $(1,1)$. So the area is

$$\begin{aligned} A &= \int_0^1 ((2x - x^2) - x^2) dx \\ &= 2 \int_0^1 (x - x^2) dx \\ &= 2 \left(\frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= 2 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - (0 - 0) \right] = \frac{1}{3} \end{aligned}$$

Example 25.3 Find the area between $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/3$.



In this case the top curve and the bottom curve switch at $x = \pi/4$. So we have to divide the solution into two integrals:

$$A = \int_0^{\pi/4} + \int_{\pi/4}^{\pi/3} = A_1 + A_2 \quad (25.3)$$

For the first integral, the top curve is $y = \cos x$ and the bottom curve is $y = \sin x$. So

$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx \quad (25.4)$$

$$= (\sin x + \cos x) \Big|_0^{\pi/4} \quad (25.5)$$

$$= \sin \frac{\pi}{4} + \cos \frac{\pi}{4} - \sin 0 - \cos 0 \quad (25.6)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \quad (25.7)$$

$$= \sqrt{2} - 1 \quad (25.8)$$

In the second integral, the top curve is $y = \sin x$ and the bottom curve is $y = \cos x$.

So

$$A_2 = \int_{\pi/4}^{\pi/3} (\sin x - \cos x) dx \quad (25.9)$$

$$= (-\cos x - \sin x)|_{\pi/4}^{\pi/3} \quad (25.10)$$

$$= -\cos \frac{\pi}{3} - \sin \frac{\pi}{3} + \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \quad (25.11)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \quad (25.12)$$

$$= \sqrt{2} - \frac{1 + \sqrt{3}}{2} \quad (25.13)$$

and the total area is

$$A = A_1 + A_2 = \sqrt{2} - 1 + \sqrt{2} - \frac{1 + \sqrt{3}}{2} = 2\sqrt{2} - \frac{3 + \sqrt{3}}{2} \quad (25.14)$$

Example 25.4 Find the area between the curves of $y = x - 1$ and $y^2 = 2x + 6$.

To see where the curves intersect we solve

$$2x + 6 = y^2 = (x - 1)^2 = x^2 - 2x + 1 \quad (25.15)$$

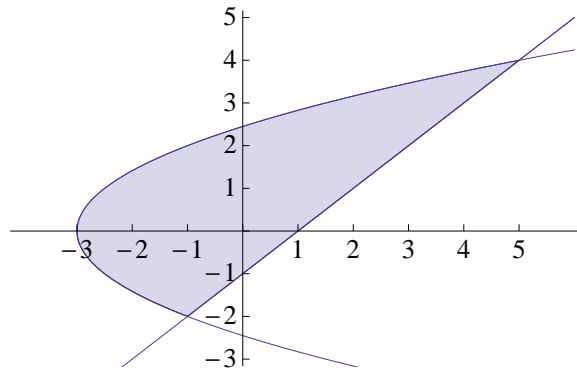
$$0 = x^2 - 4x - 5 = (x - 5)(x + 1) \quad (25.16)$$

So the intersections occur at $x = -1$ and $x = 5$.

At $x = -1$, $y = x - 1 = -2$ so the point is $(-1, -2)$.

At $x = 5$, $y = x - 1 = 4$ so the point is $(5, 4)$.

The area to be found is illustrated in the figure below.



Here again we have two different functions for the lower curve, but only on function for the upper curve.

Between $x = -3$ and $x = -1$, the upper curve is $y = \sqrt{2x+6}$ and the lower curve is $y = -\sqrt{2x+6}$. Call the area of this region A_1 .

$$A_1 = \int_{-3}^{-1} (\sqrt{2x+6} - (-\sqrt{2x+6})) dx = 2 \int_{-3}^{-1} \sqrt{2x+6} dx \quad (25.17)$$

$$= \int_0^4 u^{1/2} du \quad (\text{using } u = 2x+6 \implies du = dx;$$

$$\text{and } x = -3 \implies u = 0; x = -1 \implies u = 4) \quad (25.18)$$

$$= \frac{2}{3} u^{3/2} \Big|_0^4 = \frac{2}{3} (4^{3/2}) - \frac{2}{3} (0^{3/2}) = \frac{16}{3} \quad (25.19)$$

Between $x = -1$ and $x = 5$, the upper curve is still $y = \sqrt{2x+6}$ but the lower curve is now $y = x-1$. Call the area of this region A_2 .

$$A_2 = \int_{-1}^5 (\sqrt{2x+6} - (x-1)) dx = \int_{-1}^5 \sqrt{2x+6} dx - \int_{-1}^5 (x-1) dx \quad (25.20)$$

But

$$\int_{-1}^5 \sqrt{2x+6} dx = \frac{1}{2} \int_4^{16} u^{1/2} du = \frac{1}{3} u^{3/2} \Big|_4^{16} = \frac{1}{3} (16^{3/2} - 4^{3/2}) = \frac{56}{3} \quad (25.21)$$

and

$$\int_{-1}^5 (x-1) dx = \frac{1}{2} x^2 - x \Big|_{-1}^5 = \left(\frac{25}{2} - 5 \right) - \left(\frac{1}{2} - 1 \right) = 6 \quad (25.22)$$

Hence the total for A_2 is

$$A_2 = \frac{56}{3} - 6 = \frac{56 - 18}{3} = \frac{38}{3} \quad (25.23)$$

and the total area is

$$A = A_1 + A_2 = \frac{16}{3} + \frac{38}{3} = \frac{54}{3} = 18 \quad (25.24)$$

It turns out that there is an easier way to do this example.

Instead of using

$$A = \int_{\text{left}}^{\text{right}} (\text{Top}(x) - \text{Bottom}(x)) dx \quad (25.25)$$

we can find the area from the left curve to the right curve and integrate over y :

$$a = \int_{\text{bottom}}^{\text{top}} (\text{Right}(y) - \text{Left}(y)) dy \quad (25.26)$$

If we solve the example this way, the curve $y = x - 1$ is always on the right, and the curve of $y^2 = 2x + 6$ is always on the left.

Solving for x , the left curve is $x = \frac{1}{2}y^2 - 3$; and the right curve is $x = y - 1$. The bottom limit is then $y = -2$, and the top limit is $y = 4$. Hence the area is

$$\begin{aligned} A &= \int_{-2}^4 \left(y + 1 - \frac{1}{2}y^2 + 3 \right) dy \\ &= \left(\frac{1}{2}y^2 + 4y - \frac{1}{(2)(3)}y^3 \right) \Big|_{-2}^4 \\ &= \left(\frac{1}{2}(16) + (4)(4) - \frac{64}{6} \right) - \left(\frac{1}{2}(4) - (4)(2) + \frac{8}{6} \right) \\ &= 8 + 16 - \frac{32}{3} - 2 + 8 - \frac{4}{3} = 30 - \frac{36}{3} = 30 - 12 = 18 \end{aligned}$$

Topic 26

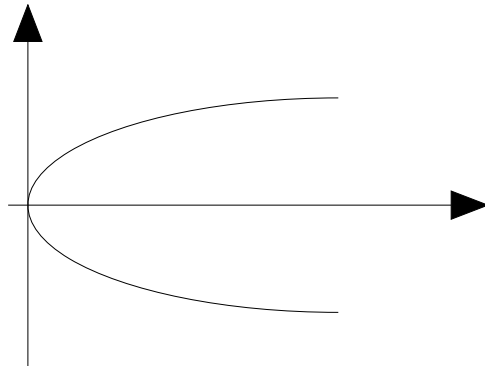
Volumes of Rotation

The idea behind a volume integral is the same as the idea behind an area integral. To compute the area, we break the area into small rectangles and add up the areas of each rectangle.

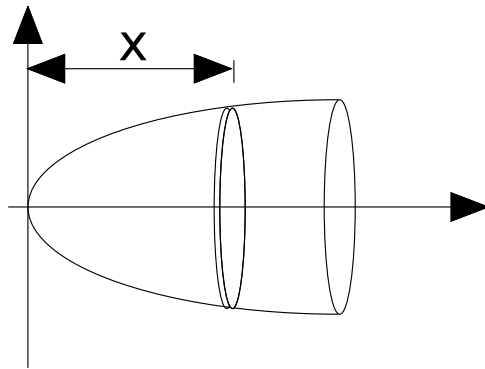
To compute a volume, we break the object up into small “slabs and add up the areas of the slabs. With some luck we can find a slab that has an area we know.

The method is illustrated in the following example.ab

Example 26.1 Find the volume of the object formed by rotating the curve $y = \sqrt{x}$ about the x axis, between the origin and $x = 1$. The curve of $y = \sqrt{x}$ is a parabola centered on the x -axis.



If we rotate it about the x-axis, it looks like this:



Consider the thin disk-shaped shaped slab oriented perpendicular to the x - axis at x that is shown in the figure. Suppose the thickness of the slab is dx . The slab has a radius equal to the value of the parabola at that point

$$r = y = \sqrt{x} \quad (26.1)$$

and hence its volume, which we call dV , is given by

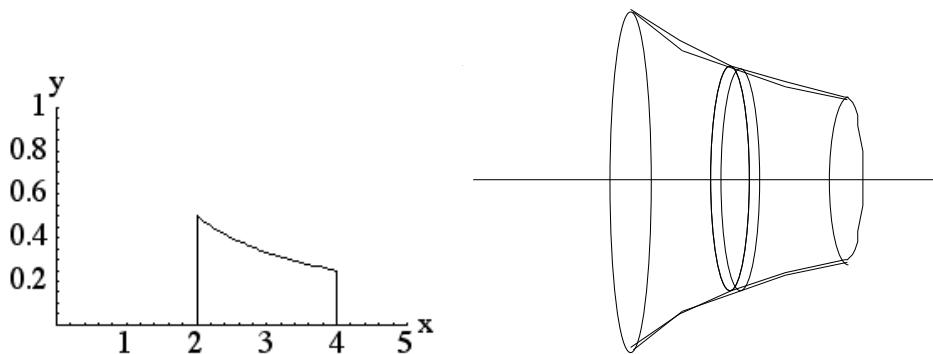
$$dV = \pi r^2 dx = \pi y^2 dx = \pi (\sqrt{x})^2 dx = \pi x dx \quad (26.2)$$

The volume of the entire object between $x = 0$ and $x = 1$ is the sum of all the slabs:

$$V = \sum dv = \int_0^1 dV = \int_0^1 \pi x dx = \frac{1}{2} \pi x^2 \Big|_0^1 = \frac{\pi}{2} \quad (26.3)$$

Example 26.2 Find the volume of the object obtained by rotating the following area about the x -axis: the region bounded by the curves $y = 1/x$, $x = 2$, $x = 4$, and $y = 0$.

The area to be rotated is illustrated in the following figure:



The volume of the rotated object, shown on the right, above, is obtained by summing the areas of the disk from the left ($x = 2$) to the right ($x = 4$)

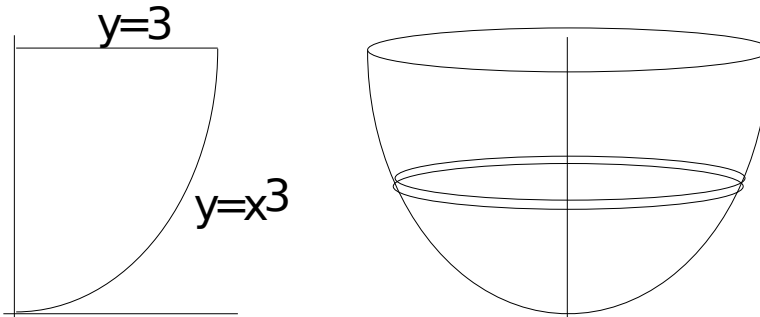
The radius of the disk at x is $r = 1/x$, so its volume is

$$dV = \pi r^2 dx = \frac{\pi}{x^2} dx \quad (26.4)$$

Hence the total volume is

$$A = \int_2^4 \frac{\pi}{x^2} dx = -\frac{\pi}{x} \Big|_2^4 = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} \quad (26.5)$$

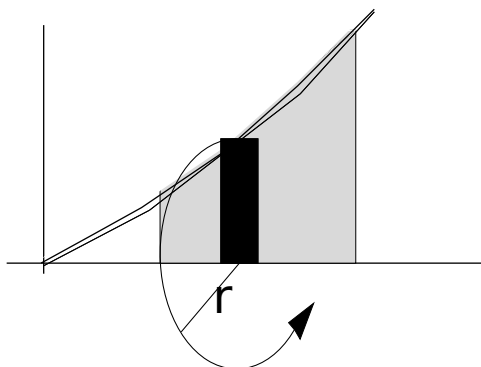
Example 26.3 Find the volume of the solid generated by rotating the region bounded by curves $y = x^3$, the y -axis, and the line $y = 3$ about the y -axis.



Since the region is rotated about the y axis we need to choose a slab that is perpendicular to the y axis, as illustrated. The radius of the disc is the distance between the y axis and the value of the x at y . Since $y = x^3$, this is the distance along the x -axis between the y -axis and the value of x at $y = x^3$, i.e., $x = y^{1/3}$. So $r = y^{1/3}$, $dV = \pi y^{1/3} dy$, and

$$V = \int_0^3 \pi y^{1/3} dy = \frac{y^{4/3}}{4/3} \Big|_0^3 = \frac{3}{4} (3)^{2/3} = \frac{3^{5/3}}{4} \quad (26.6)$$

Example 26.4 Find the volume of the object obtained by revolving the area bounded by $y = x^{3/2}$ and $y = 0$ between $x = 2$ and $x = 3$ about the x -axis.



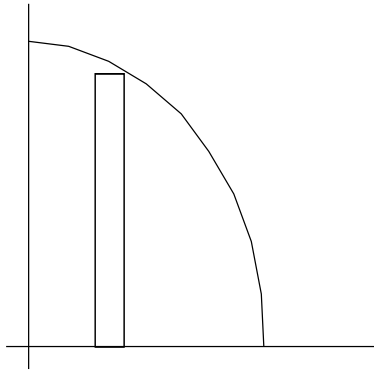
From the figure we see that the slab formed by revolving around the x -axis has radius $r = x^{3/2}$, thickness dx , and therefore the volume element is

$$dV = \pi r^2 dx = \pi x^3 dx \quad (26.7)$$

so that

$$V = \int_2^3 \pi x^3 dx = \frac{\pi x^4}{4} \Big|_2^3 = \pi \left(\frac{3^4}{4} - \frac{2^4}{4} \right) = \pi \left(\frac{81 - 16}{4} \right) = \frac{65\pi}{4} \quad (26.8)$$

Example 26.5 Find the volume of the object obtained by rotating the region in the first quadrant between the curve of $y = 4 - x^2$ and the coordinate axes about the x -axis.



The parabola intersects the x -axis when $y = 0$ or $4 - x^2 = 0$; hence $x = 2$ (we ignore the other root as it is not in the first quadrant).

At $x = 0$ we have $y = 4$.

The vertical slice at x that needs to be rotated has height $4 - x^2$; hence a rotated slab at x has

$$dV = \pi(4 - x^2)^2 dx \quad (26.9)$$

giving a total volume of

$$V = \int_0^2 \pi(4 - x^2)^2 dx = \int_0^2 \pi(16 - 8x^2 + x^4) dx = \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right)_0^2 \quad (26.10)$$

$$= \pi \left(16(2) - \frac{(8)(8)}{3} + \frac{32}{5} \right) = \pi \frac{(32)(3)(5) - 64(5) + 32(3)}{(3)(5)} = \frac{256}{15}\pi \quad (26.11)$$

Example 26.6 Find the volume of the object obtained by rotating the area bordered by the following curves about the x -axis: $y = x^3$, $x = 3$, $y = 0$.

We can break this down to vertical disks centered on the x -axis of radius x^3 between

$x = 0$ and $x = 3$. So the volume is

$$V = \int_0^3 \pi(x^3)^2 dx = \frac{\pi x^7}{7} \Big|_0^3 = \frac{3^7 \pi}{7} = \frac{2187\pi}{7} \quad (26.12)$$

Example 26.7 Find the volume obtained by rotating the curve $y = +\sqrt{a^2 - x^2}$ about the x -axis, where a is a constant.

This is the upper half of a disk of radius a . It intersects the x -axis at $x = a$ and $x = -a$.

The height at any point x is $\sqrt{a^2 - x^2}$.

A rectangle at x has height $\sqrt{a^2 - x^2}$ and width dx .

The disk formed by rotating the rectangle of width dx and height $\sqrt{a^2 - x^2}$ has volume

$$dV = \pi r^2 dx = \pi(\sqrt{a^2 - x^2})^2 dx = \pi(a^2 - x^2) dx \quad (26.13)$$

The volume is thus

$$V = \int_{-a}^a \pi(a^2 - x^2) dx = \pi \left(a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^a \quad (26.14)$$

$$= \pi \left((a^3 - a^3/3) - ((-a)^3 - (-a)^3/3) \right) \quad (26.15)$$

$$= \pi \left(a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right) \quad (26.16)$$

$$= \frac{4}{3} \pi a^3 \quad (26.17)$$

which is the expected formula for the volume of a ball of radius a .